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From action to symbols: giving meaning to the symbolic representation of the distributive law in primary school

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Abstract
The use of artifacts to introduce the distributive law of multiplication over addition in primary school is a diffused approach: it is possible to find pre-constructed learning trajectories in instructional materials. However, it is still unclear how the teacher might support his/her students in transitioning from concrete to symbolic representations of the distributive law. In the theoretical frame of the Theory of Semiotic Mediation, we report on a study where Laisant's table, an artifact embodying the rectangular model of multiplication, is used to introduce distributive law in second grade. Taking a microanalytical approach, we show how a group of students connects the representation provided by the artifact with the symbolic representation of the arithmetic property (as equivalence of numerical sentences). Two different semiotic chains are identified and presented, showing the continuity between the activity with the artifact and the mathematical signs emerging in following activities and promoted by tasks specifically designed. The role of the teacher in triggering and scaffolding this process is highlighted.

Keywords Distributive law · Multiplication · Semiotic chain · Semiotic mediation · Primary school · Rectangular model

1 Distributivity in primary school

Distributive law (DL) has a strong importance in algebra: it defines the structure of field, and it is basilar for studying polynomials and vector spaces. Also, it is a property characterizing the operations between whole numbers, and so it might be faced since the first introduction of addition and multiplication. According to Carpenter, Levi, Franke, and Zeringue (2005), "an

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implicit understanding of the distributive property can provide students a framework for learning multiplication number facts by relating unknown facts to known facts" (p. 55). Hence, awareness of the distributive relationship between sum and multiplication may be useful before memorizing multiplication facts, early in primary school. Also, DL results as fundamental for avoiding an only procedural understanding of multiplication and division algorithms as they are thought at primary level (Izsák, 2004).

The symbolic representation of DL seems hardly accessible to young children (Linchevski & Livneh, 1999); thus, DL is often introduced by graphical representations (Ding & Li, 2014; Izsák, 2004). Freudenthal claims that "only by means of the rectangle model of the product do properties of multiplication become visible: […] distributivity, two rectangles of equal height (or width) moved side by side" (Freudenthal, 1986, p. 26). Even if the "only" in the beginning of this last quotation could be controversial (Larsson, 2015), there is more and more research supporting that the rectangular model is suitable to foster argumentations about DL (Barmby, Harries, Higgins, & Suggate, 2009; Freudenthal, 1973), and introducing distributivity by this model at the very beginning may help in the future years of schooling (for a broader review, see Maffia & Mariotti, 2018).

Ding and Li (2014) found that Chinese textbooks propose different representations of the DL and present tasks purposely designed for the transition from concrete representations (related to artifacts or word problems) to more abstract ones. They conclude that studying how instructional materials are then implemented is relevant to better understand how this transition happens. The introduction of mathematical contents through manipulatives and other artifacts is a widespread practice in primary schools. However, decontextualization of the mathematical content from the context of the activity with the artifact and connection of the artifact with other representations of the same content (e.g., mathematical symbols) cannot be considered as spontaneous for students (Bartolini Bussi & Mariotti, 2008). Introducing DL through the rectangular model, mathematics teachers expect that students will relate the areas of rectangles and the combinations of rectangles to multiplications and additions respectively; we wonder how this may happen and, in particular, how graphical representations are then linked to symbolic ones. We take a semiotic approach to consider what Ding and Li (2014) call the "transition from concrete to abstract" as the shifting from activities realized using an artifact (Laisant's table, see Section 3) to the symbolic representation of DL as equivalence of number sentences.

2 Theory of Semiotic Mediation

An artifact is defined as a *tool of semiotic mediation* when the teacher uses it intentionally to mediate a mathematical content to students (Bartolini Bussi & Mariotti, 2008). In our case, we focus on the use of Laisant's table by the teacher to mediate the symbolic representation of DL. According to the Theory of Semiotic Mediation (TSM), the artifact is related both to the personal signs produced by its user while solving a task and to the mathematical knowledge underpinning the task and/or artifact itself. Adopting Eco's (1973) interpretation of the Peircean definition, we consider a sign as "something which stands *to somebody* for something in some respect or capacity" (p. 27, emphasis added).

The double relation of the artifact with the accomplishment of a task and the underpinning mathematics is called *semiotic potential* of the artifact (Bartolini Bussi & Mariotti, 2008). It is studied *a priori* by making explicit such relations: in Section 3, this kind of analysis is presented for the artifact used in our experiment. When expected artifact signs emerge, we say that the *semiotic potential* of the artifact has *unfolded*. Indeed, when the artifact is used, signs referring to the context of the artifact (to one of its parts and/or to the action accomplished with it) are produced. Those signs, to which we refer as *artifact signs*, can be different from those used by mathematicians while working with the mathematical knowledge related to the task. However, according to TSM, an evolution from artifact signs to *mathematical signs* is not only desirable, but rather is the aim of the process of semiotic mediation, fostered by the teacher. In order to describe such evolution, we use the notion of *semiotic chain*; it is the set of dynamic relations among artifact signs and mathematical ones. Two signs are considered as linked when one is used to interpret the other, for instance, if a word is used to describe a drawing or a gesture. When a new sign replaces or is linked to a previous one, we can follow a development process in which signs change and (desirably) the meanings initially related to the artifact converge to those shared by the mathematical community (Sáenz-Ludlow, 2006).

Through a mathematical discussion, the teacher can foster students' personal signs and connect them to mathematical signs as expected in the a priori analysis (Fig. 1). According to a Vygotskian perspective, learning is promoted by a more expert one acting in the student's zone of proximal development; in TSM "the teacher acts as mediator using the artifact to mediate mathematical content to the students" (ibidem, p. 754). The teacher observes students' productions and selects specific signs with the intention of fostering the unfolding of the potential of the artifact (Maffia, 2019). These signs are defined with respect to their function in promoting the relationship between the other two categories of signs (Bartolini Bussi & Mariotti, 2008, p.756). They are called *pivot signs* because of their role in moving from artifact signs to mathematical ones. Pivot signs may be *hybrid* in their nature, namely, referring at the same time to the artifact and to mathematics.

Signs in semiotic chains can be words, inscriptions, and drawings, but they can come by other *semiotic sets* (Arzarello, 2006). For instance, Maffia and Sabena (2016) show how gestures can be part of a semiotic chain, and so they speak about multimodal semiotic chains. The multimodality can be described by the presence of *a semiotic bundle*, namely:

> a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a

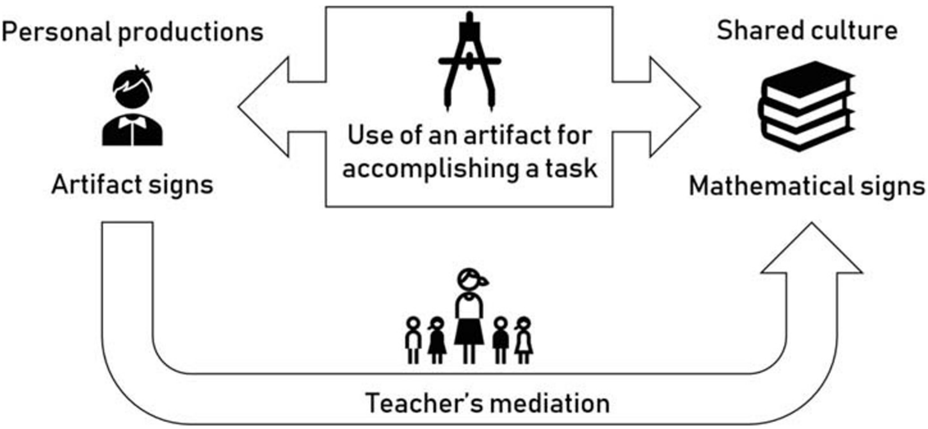


Fig. 1 Semiotic potential of an artifact and the teacher's mediation

mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello, Paola, Robutti, & Sabena, 2009, p. 100).

Referring to the presented framework, we analyze the signs produced by students and teacher for investigating the semiotic mediation process triggered by the use of Laisant's table. We formulate the following research question: What are the signs, and specifically the pivot signs, constituting the semiotic chain linking artifact signs emerging from the use of Laisant's table to the symbolic expressions of the DL?

3 The semiotic potential of Laisant's table

In order to support the complexity of symbolic representation of operations' properties (Linchevski & Livneh, 1999), graphical models are often proposed (Ding & Li, 2014; Izsák, 2004), and among others, the rectangular model is widely considered preferable (Freudenthal, 1973; Barmby et al., 2009; Maffia & Mariotti, 2018). For instance, when put one next to the other, two arrays of marbles with the same number of rows form another array; however, the nature of this array, though allowing students to manipulate each item of an array, easily rearranging the marbles into different arrays, does not allow moving arrays of marbles as a single unit, e.g., preserving the global arrangement through a single movement. On the opposite, rectangles cut from a sheet of paper are manageable as wholes and allow an enactive manipulation of the whole multiplication.

In his book *Initiation mathématique* …, the French mathematician Charles-Ange Laisant introduces "a times-table without any digit" (Laisant, 1915, p. 47). His original drawing is shown in Fig. 2. This table is also known as "decanomial" in Montessori's (1934/2016) tradition. In Laisant's table, columns width increases by one going from left to right, and rows height also increases by one from top to bottom. Each cell is a rectangle and represents a multiplication: the lengths of sides are the factors, and the area is the result. Laisant's table incorporates the rectangular model, presenting any rectangle as an ordered multiplication. Such possibility constitutes the core of the semiotic potential of this artifact.

Specific tasks were designed to relate the rectangular representation embedded in Laisant's table to the DL. The first task asks students to cut two pieces of paper so that each piece matches exactly one of two cells in the same row (both sides smaller than five). They also must paste the pieces of papers along one side to form a new rectangle; then, they are asked to look for a rectangle with the same dimensions in the table. Children are expected to drag and rotate the paper rectangle, trying to fit it inside one of the cells. We also expect pupils to notice that all the three involved rectangles have the same height and the final rectangle has a width that is the sum of the two others; this constitutes the germ of the mathematical meaning of the DL. According to the interpretation of any rectangle as a multiplication, the action of pasting two rectangles can be interpreted as the addition of two multiplications, so the new rectangle can be interpreted both as a multiplication and as the sum of two multiplications. In other words, pasting two rectangles represents the transformation of an arithmetical expression according to the DL: $a \times b + a \times c \rightarrow a \times (b + c)$.

In order to work on the other direction of the transformation ($a \times b + a \times c \leftarrow a \times (b + c)$), another task is designed: students received a letter from Giovanni, an imaginary child. Giovanni explains that he must calculate 3×7, but he only remembers

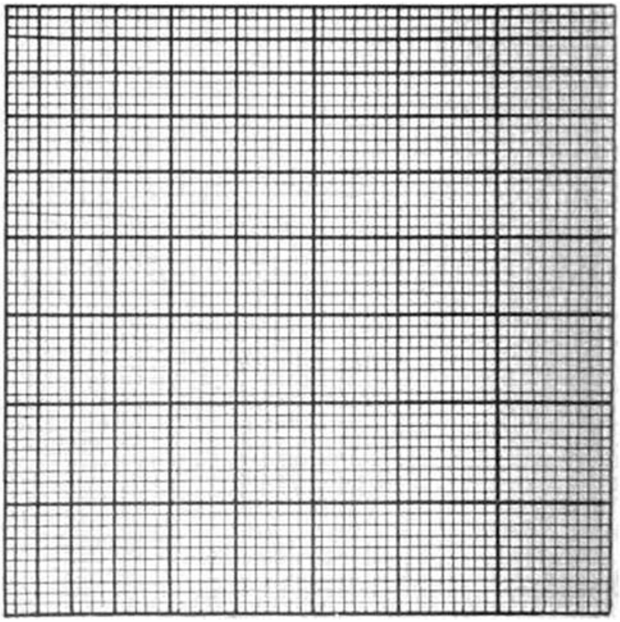


Fig. 2 Laisant's table

multiplications with factors smaller than five. We expect children to refer to the past activities (to the signs emerged in that occasions) and decompose 3×7 in smaller multiplications, using the table to find the right ones.

The combination of these two tasks leads students to work both on composing and decomposing tiles and related multiplications. The emergence of artifact signs lays the base for the collective discussion and is expected to prepare the evolution toward the symbolic expression of the DL.

4 Methodology

The data come from a design study aimed at developing tasks for introducing the rectangular representation of multiplication and subsequently representing the operation properties as equivalences between number sentences. We follow the paradigm of design-based research (Bakker & van Eerde, 2015; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003): a teaching sequence is designed, by a team of teachers and researchers, following the a priori analysis of the semiotic potential, serving as hypothetical learning trajectory (Bakker & van Eerde, 2015) in the phases of design, implementation, and analysis. The tasks are administered, and the consequent activity is analyzed both with the aim of refining the tasks and answering theoretical questions as the research question of this paper.

The excerpts presented below come from the first implementation of the tasks, and the presented analysis is part of the retrospective analysis (Bakker & van Eerde, 2015). They refer to a grade 2 class group of 20 students aged 7–8. In the first weeks of the intervention, children explored the artifact, and they were asked to imagine how numbers and operations could relate to it. After 2 months, the tasks shown in previous section were presented.

The tasks were designed and implemented with the teacher. She is experienced, but she never used Laisant's table before. The theoretical constructs presented above were not known to her, but the aim of relating students' personal productions to symbolic representation of DL was shared between the researchers and the teacher. The students worked individually on a Laisant's table of five rows and five columns. A table with ten rows and columns was available on the wall of the classroom. The first author and the teacher interacted with students during individual activities. Collective discussions were co-orchestrated by them after sharing the objectives of the intervention.

Lessons have been videotaped. The camera was not fixed, so it was possible to move it during collective discussions to frame the speakers (the students, the teacher, or the researcher). Students' written productions were captured using a camera. Additional information about the teacher's interventions and her intentions were collected by interviewing her.

Following the analytical model for studying videotape data by Powell, Francisco, and Maher (2003), videos have been viewed several times, and a description of the events was written in a narrative form. According to a priori analysis of the semiotic potential, *critical events* were selected. We selected all those episodes in which a new sign is introduced. This is coherent with the idea of semiotic node presented by Radford and Sabena (2015). Critical events are fully transcribed and coded. Screenshots of gestures and inscriptions are added to the transcript (as described by Radford & Sabena, 2015).

We adopt a microanalytical approach (ibidem): using the TSM lens, we analyze the individual activity of children while facing tasks involving Laisant's table and the collective discussions. According to TSM, our units of analysis are signs. In order to describe the process of semiotic mediation, we categorize different types of signs: artifact signs, mathematical signs, and their relations. We identify pivot signs and so describe the development of the semiotic mediation process. Specifically, we try to outline semiotic chains showing the move from personal to mathematical sings.

In the following section, selected episodes are analyzed. Each line is numbered according to the original transcription. Children's names are replaced with pseudonyms.

5 Toward the symbolic expression of the DL: analysis of the semiotic mediation process

In this section, we present a sequence of episodes, each one in a different subsection. These activities do not represent the first meeting of the children with Laisant's table. Children were introduced to the table by letting them observe and describe the regular increment in the height of rows and the width of columns. Such activity of description prompts the generation of signs that can be related to the different parts of the table. In particular, the word "tile" was proposed by students to refer to the cells. The reproduction of the table – combined with cutting and moving rectangles and recognizing them on Laisant's table – allowed students to observe a relation between the lengths of the sides of each cell and the number of squares inside it. The teacher proposed to represent such relation according to the usual mathematical symbols for multiplication, so they can relate multiplications to rectangles and vice versa. At this point, a shared system of artifact signs is established around the keyword "tile" and its meaning: it refers to one of the rectangles either on the table or cut on the paper but also to the multiplication between the two numbers representing the dimensions of the rectangle. Thus, because its meaning combines both the reference to the artifact and the reference to mathematics, the sign "tile" has the potential of being a pivot sign.

5.1 Combining tiles of the same row interpreted as adding

The first episode is a part of the discussion following students' work on the first task described in section 3. They had to select two tiles in the same row of the table, to cut them and to combine them into a new tile, and to identify on the table a rectangle corresponding to this new tile. When the task was completed, the teacher opened a collective discussion. A child (Lor) makes the first intervention:

1 Teacher: Were you able to find the big tile? What did you see?

2 Lor: That...when two tiles are far away (he points at the table with ten rows and columns that is on the wall), you can calculate the result and then you know it.

3 Researcher: And how do you calculate the result?

4 Lor: Between these two (he points two rectangles in the fourth row), you do nine times four; it is thirty-six (he points the 4 × 9 rectangle[1]) plus twenty (he points the 4 × 5 rectangle); it is fifty-six.

5 Researcher: […] Well, Lor gave an example, and he said that when the results of two tiles are known, it is possible to discover the result of another tile. I have understood this way; you have to say to me if I understood correctly. You said that if I know the result of two tiles (he does the gesture in Fig. 3a), I can do the addition (gesture in Fig. 3b). Isn't it?

6 Lor: No. It is that if you do these two that are far (he points the two rectangles), you calculate them!

Lor intervenes in the discussion with a sentence involving and relating both artifact signs and mathematical ones (line 2). He is speaking about tiles, but then he refers to calculating the result. The interpretation of the word "tile" as representing an arithmetical operation that has its own result is clear. It is unclear what is the operation to which Lor is referring. The researcher elaborates Lor's statement generalizing it and passing to the interpretation of the calculation – addition of multiplications – as the combination of tiles (line 5). The semiotic process of interpretation is accomplished by an enchainment of signs of different nature: words, graphical representations, mathematical symbols, and gestures (Fig. 3). The word "addition" (a mathematical sign) is combined with the gesture of joining the fingers (a gesture related to the action performed with the artifact), with the intention of relating the combination of tiles (represented by the gesture) and the operation of adding numbers. Lor seems not to appreciate the researcher comment, maybe because it does not stress what is important for him: the fact that he is referring to tiles which are *far* from each other (line 6). Apparently, this child is referring to the problem of finding the results of the combination of two tiles which cannot be easily imagined one next to the other (because of their distance). We can also notice that he is producing a new example (line 4), different from the one he analyzed during the individual activity (indeed numbers are greater than five), and he does it without cutting new pieces of paper. Hence, he is generalizing the previous activity to new examples, but the reference to DL is still not explicit. The intervention of just one child does not mean that all the children reached the same generalization, so there is the need of covering the gap between Lor's

[1] The Italian reading of multiplications is different from the usual English one. In Italian, 4 × 9 is read as "quattro per nove," meaning four repeated nine times.

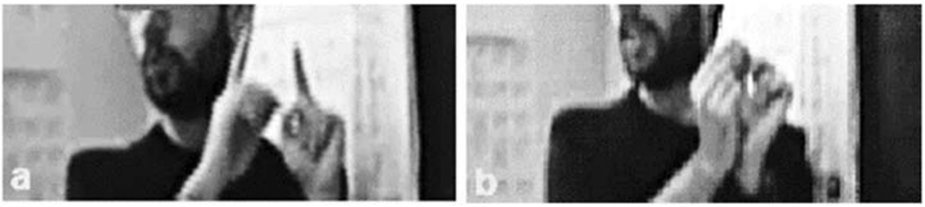


Fig. 3 Researcher's gestures during discussion

generalization and what could be expected from the other pupils. In the next excerpt, the researcher continues the discussion trying to work on a simpler example to come back to the task and make Lor's intervention understandable for the classmates.

7 Researcher: What do you obtain?

8 Lor: Fifty-six.

9 Researcher: And what do you need this number for? What does it represent?

10 Lor: A tile.

11 Researcher: It is another tile; it is what I was saying: if you put together two tiles, then you find another one. Let us do an example […] if I, for instance, take the tile three times two (he draws a 2 × 3 rectangle on the blackboard, he writes "2 × 3" inside it). Do you all agree that this is the three times two tile?

12 Chorus: Yes!

13 Researcher: And I put together the four times two tile (he draws a second rectangle juxtaposed to the previous one, Fig. 4a) this is as big as which tile? Putting all together?

In the elaboration of the new example, the mathematical sign "addition" (used by the researcher in line 5) is replaced with the artifact sign "put together" (lines 11 and 13) referring directly to the activity done (i.e., the pasting of the pieces of paper) with the aim of bridging the gap between the idea of combining the tiles and that of adding multiplications. The researcher also introduces a graphical icon by drawing rectangles with multiplications inside (Fig. 4a), and he accompanies his inscriptions with oral signs: the researcher refers (line 11) to this graphical representation using the word "tile", an already shared artifact sign, and reading the multiplication written inside, a mathematical sign. We can notice that the researcher focuses the pupils' attention on this new icon, and he wants it to be shared; indeed, he asks: "Do you all agree that this is *the three times two tile*?" (line 11). The fact that the children approve (line 12) suggests that the new meaning related to the word "tile" is shared. Those meanings already associated with the word "tile" can now be related to the introduced graphical representation.

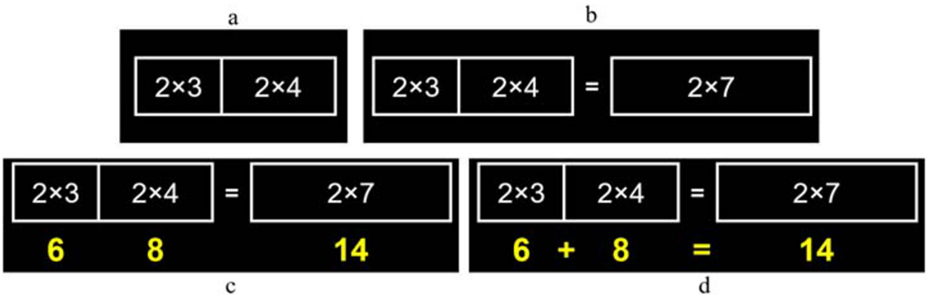


Fig. 4 Reconstructions of inscriptions on the blackboard

The emerging semiotic bundle allows the construction of a system of relationships between signs of different nature – iconic artifact signs and mathematical symbols, written and oral – produced and shared by different actors. According to Lor, "fifty-six" is a "tile" (line 10), so the result of the multiplication is identified as a possible interpretation of a tile. The researcher relates the word "tile" to a multiplication referring to the "tile three times two" (line 11), so the word "tile" is used both as the operation and its result at the same time. We can recognize the development of a semiotic chain that is realized through a genetic conversion (Arzarello, 2006), namely, the oral sign (tile) and gestural signs (Fig. 2), which are converted on the blackboard in new graphical ones (Fig. 4a).

After the children have agreed on the answer to the last question (line 13), the researcher synthesizes the interventions saying that the two tiles, together, equal[2] the three times seven tile, and he draws it (Fig. 4b). Finally, the teacher asks the children to say how many squares are contained in each rectangle. Children say the numbers, and the teacher writes them (in yellow) under the drawing (Fig. 4c).

Some students notice that six plus eight is fourteen. The researcher decides to rephrase one of his previous sentences.

29 Researcher: So, when I put together the squares inside this tile (he points the 2 × 3 tile) with the squares of this tile (he points the 2 × 4) I obtain the squares of this whole tile (pointing 2 × 7). Isn't it?

30 Non: It is true!

31 Researcher: Which operation does "put together" correspond to?

32 Chorus: Six plus eight!

33 Mab: Equals fourteen.

34 Researcher: (writes the symbol + and the symbol = between the numbers, Fig. 4d) So, what does it mean? If I know the results of two little tiles (he points 2 × 3 and 2 × 4), I can put them together (he points the numbers 6 and 8) and what do I find? (he points at 14) The result...

35 Chorus: Of a tile!

36 Researcher: And how do we find this tile? It has the same height and this? (he points at the base of the rectangle) …

37 Sim: It is as large as the two together.

38 Researcher: It is as large as the two together. Do you all agree?

39 Chorus: Yes!

The researcher acts as the expert mathematician who sees the potential of the word "put together" (already used in lines 11 and 13) to relate the activity of pasting rectangles to the operation of addition. In order to construct the chain toward the mathematical signs, he speaks about putting together tiles to obtain tiles (line 13). Then, once the word "tile" is related both to the multiplications (by the researcher) and to their results (by Lor), the operation of putting tiles together can be interpreted both as adding multiplications or adding their results.

In this last excerpt, the sign "put together" works as pivot sign: firstly, it is interpreted with reference to the action of combining the tiles. Afterwards, the interpretation requested by the researcher and the elaboration of the complex signs mixing the icons and the symbols leads the expression "put together" to be explicitly related to the operation of addition.[3] Indeed, when

[2] The phrase used by the researcher is "è uguale." In Italian, it has different meanings according to the linguistic register. It means "it equals" in the formal-mathematical register, and it means "it looks the same as" in colloquial register.

[3] Likely, the association of the metaphor of putting together to the operation of adding was already known by the students. Such metaphor is common in Italian textbooks for first grade.

the researcher asks the children to interpret “put together” (line 31), the students link this sign to the operation of adding and express this interpretation with the verbal sign “plus” (line 32), then converted in a graphical sign (the symbol + in line 34, Fig. 4d). Again, we have a conversion of a verbal sign into an inscription that fosters the recognition of the additive relation between the numbers 6, 8, and 14. Mab, proposing the result of the addition (line 33), implicitly suggests the relation between the operation of putting together and the result of the addition. The interpretation process goes on, and when the researcher points at the result of the sum (line 34), the children interpret it as the result of a tile (line 35), apparently referring to the one obtained by pasting rectangles. Indeed, Sim states that the tile is as large as the other two together (line 37). The researcher asks the other students if they agree, and their answers suggest that this meaning is shared.

5.2 Sum between rectangles and sum between results

At the end of this discussion, the children are asked to write down (even by drawing) combinations of tiles different from those discussed collectively. This semiotic activity allows to observe different ways of representing the sum of multiplications (the pasting of tiles) which recall (in a more or less iconic way) the experience with slips of paper, but at the same time the different representations shared during the discussion. Two children cannot accomplish this task. The others give one or more original examples imitating the hybrid signs of Fig. 4c to represent the DL (e.g., Fig. 5b). Four children add the sign + between the rectangles, so creating by themselves, individually, a new representation (Fig. 5a) that was never used before. All these texts are made of two lines: the first line is composed of hybrid signs and the second one of mathematical signs. For instance, in the drawing reported in Fig. 5b, we have a first line representing rectangles (artifact signs) containing a symbolic representation of multiplications inside. The numbers written inside correspond to the dimensions of the rectangles, so there is a relation between the multiplication written inside and the drawn rectangle, showing that the pupil make a personal use of the signs shared in the previous discussion (line 11). Furthermore, each rectangle is associated with the result of the multiplication written inside and in some cases also to the number of squares (Fig. 5a), imitating the inscription produced during the discussion (Fig. 4). The additive relation between the results is expressed by using the symbols + and =, as suggested by children’s words (line 32–33). In Fig. 5a, such additive relation is observable also between the rectangles. Children proposing this kind of representation are interpreting the symbols + and = as referring to the operation of putting together the rectangles (an interpretation suggested during the discussion, line 11), but at the same time, they are relating this meaning to the mathematical meanings, as shown by the second line. Consistently, in Fig. 5a, we can interpret the bold line, appearing in the rectangle at the right side of the equal sign, as aimed at stressing that the rectangle 3×4 is the result of the pasting of other rectangles. The text in Fig. 5a presents hybridization between the different semiotic systems providing evidence of the movement from contextual meanings, strictly related to the use of concrete

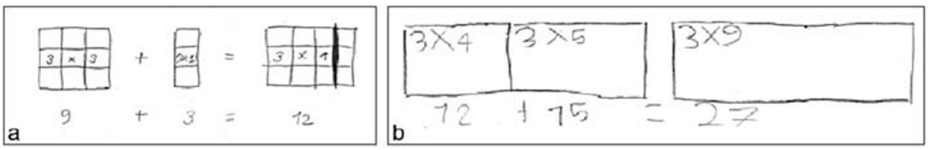


Fig. 5 Fra’s (**a**) and Mal’s (**b**) productions after the class discussion

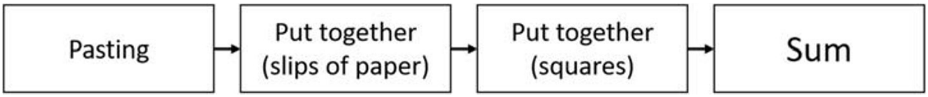


Fig. 6 Semiotic chain connecting the activity of pasting to the mathematical signs related to sum

tiles, to mathematical meanings expressing the relationship between operations. Students are replicating, at an individual level, the movement from artifact signs to mathematical ones as it happened during the collective discussion. Thus, the emerged hybrid signs worked as pivot signs.

Such movement can be described as a semiotic chain (Fig. 6) constituted of the signs emerging both in collective and individual semiotic activities. Indeed, between lines 7 and 13, the operation of pasting is described as "put together," referring to the slips of paper. In line 29, "put together" is referred to the squares inside a rectangle, and then, between lines 32 and 34, the word "squares" is replaced by the word "results" (referring to the results of the multiplications inside the rectangles). In line 32–33, the children suggest interpreting the expression "put together" as sum, represented both verbally and symbolically. In final personal productions, the symbol + can be interpreted both as the sum between the results (Fig. 5b) and as a sum between rectangles (Fig. 5a).

5.3 A Rosetta Stone for translating rectangles into number sentences

At this point, the teacher is convinced that the situation is sufficiently mature to step further toward the mathematical signs that are the aim of the teaching sequence. She proposes a variation of the individual task of representing the combination of rectangles and gives them an example of a different representation drawing it on the blackboard, the one that is reproduced in Fig. 7a. Then, she asks them to invent some other personal examples with different numbers.

Part of the representation proposed was co-constructed during the mathematical discussion presented above, but a new part is added by the teacher. A new line is put above the previous ones; like the line at the bottom, it displays a symbolic number-sentence ($3 \times 7 + 3 \times 5 = 3 \times 12 = 36$). Pupils are asked to interpret this new line – proposed in parallel with the previous ones – and then to produce personal examples with different numbers. This parallel presentation may suggest that the new line provides an alternative "paraphrase" of the other lines, containing artifact signs (rectangles) and mathematical signs. It could be considered a kind of Rosetta Stone: the archeologists had the opportunity to give a meaning to hieroglyphs, thanks to a two-line text in which the same content was provided both in the already known ancient Greek language and the unknown Egyptian hieroglyphs. In the same way, this combination of three lines proposed by the teacher establishes an explicit relationship between the inscriptions

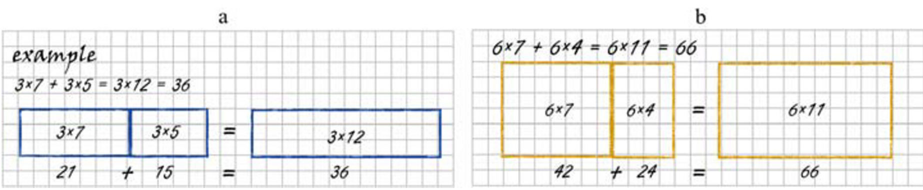


Fig. 7 Mir only copies the teacher's example (**a**) while Lor invents a personal example (**b**)

having their meanings in the previous experience and those appearing in the new line, introducing a new way of representing relations between operations. It solicits the translation of artifact signs into mathematical ones. Thus, it may foster the evolution of signs fueling a semiotic chain.

Though at this moment only a few students could produce texts including lines written in pure mathematical signs (Fig. 7b), the availability of a translation key from a semiotic system to another opened the way to the following activity.

5.4 Arithmetical symbols to represent a number sentence

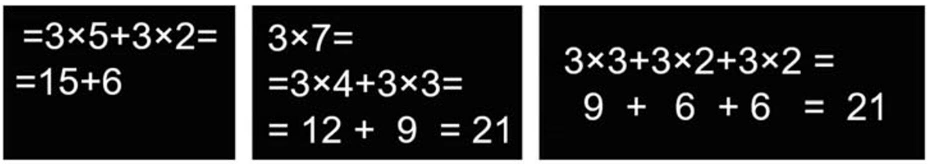
As described in Section 3, a few days after, the pupils received a letter written by an imaginary child asking for help: he must calculate 3×7, but he only remembers multiplications with factors smaller than five. Students must discuss possible answers to Giovanni in small groups and then report their solutions to the whole class group. Proposed solutions consisted of representing the multiplication as a rectangle and then decomposing the rectangle 3×7 in two smaller rectangles 3×4 and 3×3, or 3×5 and 3×2. The pupils elaborated the decomposition representing it with the graphical hybrid signs developed in previous activities. The collective discussion begins when the teacher asks each pupil to report about his/her own solution. The teacher invites them to the blackboard. Fra starts by writing the numerical sentence $3 \times 5 + 3 \times 2$ (using symbolic mathematical signs); then, he adds rectangles around each of the two multiplications. The rectangles have the same height, but apparently Fra does not care about their lengths.

Some children affirm that they wrote the same answer proposed by Fra, and then Mir asks to go to the blackboard because he would like to show a new solution, and he does it (Fig. 8a). This is the first time that Mir writes a pure symbolical number sentence. From the mathematical point of view, what Mir writes does not differ from Fra's proposal. The difference is only in the fact that Mir's writing eliminates any reference to the artifact, using only the symbolic register. Asked by the researcher to interpret the inscription, he answers:

Researcher: Ok, Mir, can you explain something to me? What do all the equal signs mean?

Mir: Three times five equals fifteen. Three times five plus three times two equals fifteen plus six.

Mir can interpret each part of his writing mathematically, establishing correct relationships between some of its components (three times five equals fifteen), but also as a whole, considering it as an equality between two number-sentences (three times five plus three times two equals fifteen plus six). When Mir receives positive feedback from the teacher, other



a b c

Fig. 8 Reconstructions of inscriptions on the blackboard written by Mir (**a**), Nic (**b**), and Lor (**c**)

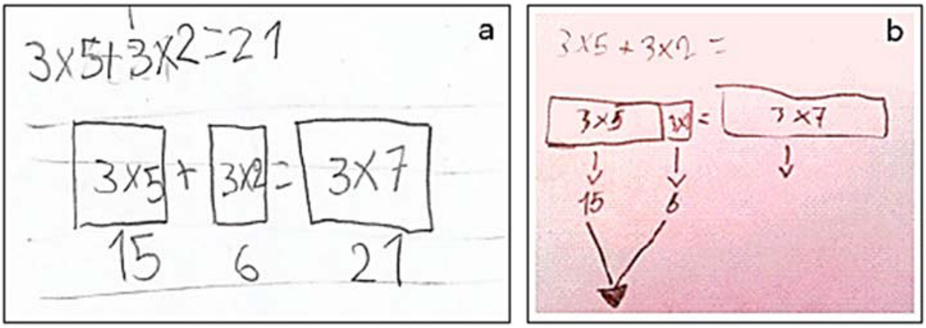


Fig. 9 Sob's and Sim's combined usage of numerical sentences and rectangles

students ask to go to the blackboard to use the same representation for different multiplications (Fig. 8 b and c).

These mathematical signs can be easily related to the signs previously used (as shown by Fra's solution), but after Mir's intervention, they can be used abandoning the iconic component referring to the artifact. At this point, the texts produced rely only on mathematical signs. The meaning of the mathematical text was not explained by the teacher; nevertheless, the linkage with previous signs, condensed in what we have called a Rosetta Stone, allows students to interpret it. The semiotic chain is now completed.

After the collective discussion on the different proposals, students are asked to write individually a letter to Giovanni. Five children's answers are expressed by pure numerical sentences, while three students use only a representation with rectangles. In many productions, the two representations appear together (Fig. 9), showing evidence of the process of linkage of artifact and mathematical signs. Possibly, it is the aim of offering an explanation that moves the children to use both the representations, as if they wanted to show to Giovanni how the symbolic inscription on the top should be interpreted.

Summarizing, these last episode shows evidence of an evolution of the signs used to represent multiplications: a second semiotic chain can be identified (Fig. 10), linking multiplications inserted in a rectangle (a hybrid sign shared in the discussion presented in Section 5.1) to multiplications expressed by arithmetical symbols. In Section 5.2, we observe that the word "tile" was used to refer both to multiplications and their results. Furthermore, the hybrid representation of a rectangle with a multiplication inside (Fig. 4) was used by children even in individual productions (Fig. 5). The Rosetta Stone presented by the teacher (Section 5.3, Fig. 7) constitutes an effective pivot providing an explicit interpretation key. The discussion about the solutions presented by Fra and Mir suggested and legitimized the possibility of using just arithmetical symbols to represent a number sentence involving multiplications.

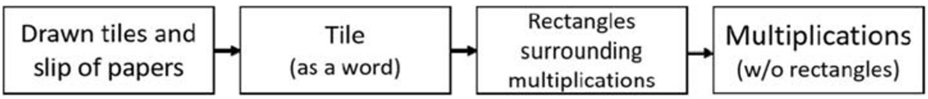


Fig. 10 Semiotic chain connecting slips of paper to the symbolic representation of multiplications

6 Discussion

While designing and analyzing tasks aimed at fostering primary students to develop symbolic representations of the DL, we must consider that "carefully designed tasks alone may not ensure a successful representational transition, just as a good vehicle cannot guarantee arriving at the destination. This is because […] students tend to make arbitrary connections between representations" (Ding & Li, 2014, p. 119). As a matter of fact, in our analysis, we show the long way that has to be covered to reach a meaningful use of mathematical signs to express arithmetical facts and how crucial is the role of the teacher: her skillful handling of the signs that emerged from students' activities was successful in keeping the symbolic representation connected to the activities realized with the artifact.

Hybrid signs belonging to different semiotic systems effectively functioned as pivot signs. The pivot sign "put together" was expressed in word but also related to a gesture recalling the metaphor of union of set of objects (Fig. 3). Similarly, the inscriptions of multiplications inside rectangles, made of signs from different semiotic sets (the drawing of the rectangle and the arithmetic symbols), showed their potential in the evolution of the discourse. These hybrid signs constitute a semiotic bundle (Arzarello, 2006); they can be related either to the artifact or to mathematical symbols; sometimes they are used in parallel or generated one from the other. Such semiotic richness, characteristic of a semiotic bundle, may explain the possibility to move from representing a combination of tiles (rectangles) toward representing a relationship between arithmetical symbolic expressions. Our results corroborate and strengthen results coming from other studies, attesting that semiotic bundles can work as pivot signs shaping multimodal semiotic chains (Farrugia, 2017; Maffia & Sabena, 2016). Signs are not simply chained one after the other, but they are used in parallel. This potential was intentionally exploited by the teacher through the production of hybrid texts explicitly relating – as a Rosetta Stone – the two different systems of signs.

The use of the linear model of semiotic chain to frame our analysis resulted into the identification of two semiotic chains, eventually leading to a symbolic representation of the distributive relation between multiplications that is very close to the one expected in the a priori analysis and has the potential to evolve toward the more general $a \times b + a \times c = a \times (b + c)$. However, our chains can be considered an oversimplification of the complex semiotic process that occurred. Indeed, the two semiotic chains are intertwined and originate a *semiotic net* (Mariotti, 2013): the availability of symbolic representations – both for multiplication and for sum of multiplications – led the children to produce symbolic sentences that can express the DL.

The identified semiotic chains help us in describing how the symbolic mathematical inscription used by the children relate to the previous activities and so are soaked by the meanings related to tiles, squares counting, rectangles areas, etc. (in brief, to the semiotic potential of Laisant's table). Though any reference to the artifact disappeared, we believe that a distillation of meanings coming from the activity with the Laisant's table will persist in pupils' knowledge.

The chosen methods allowed us to make a fine-grain analysis of the semiotic mediation process, but it was possible to do it only in a specific case: a few tasks in just one class group. For this reason, even if our results give us a precise picture of a possible teaching and learning trajectory to introduce the symbolic representation of DL, they cannot be considered as general results. Our analysis shows clearly that we observed an instance of a smooth movement from a concrete representation of DL to a symbolic one and that such movement can be realized after the intervention of the teacher acting purposefully to foster the construction of a rich semiotic net relating students' personal semiotic productions to those belonging to the mathematical culture that she represents in the classroom.

The kind of tasks proposed seems to be promising for introducing the usage of symbolic expressions in the very first grades. We do not claim that the proposed trajectory is the most desirable or better than others, but it deserves attention because of the obtained results. As far as the authors know, the semiotic potential of Laisant's table is still not fully studied (Maffia, 2019); nothing was done until now in relation to the DL. We expect that what we have observed in this case may apply to other practices involving the rectangular representation of multiplication, making our results of more general interest. Further research is needed: the implementation of new experiments will allow to get further information about the effectiveness of tasks involving such representations and about how to implement them in the classroom.

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