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Editor-in-Chief: **B Bouchon-Meunier**
Laboratoire Formes et Intelligence Artificielle
Université Paris VI, Case 169
4 Place Jussieu, F-75252 Paris Cédex 05, France
Tel: 33-1-44-27-70-03
Fax: 33-1-44-27-70-00
e-mail: bouchon@laforia.ibp.fr

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Enclosed please find the comments of the referees regarding your paper # 96/18 entitled

Multiple Valued Discrete and Continuous (Fuzzy) Logics by H.M. Hubey

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Multiple Valued Discrete and Continuous (Fuzzy) Logics

H.M. Hubey, Associate Professor
Department of Mathematics, Computer Science, and Physics
Montclair State University
Upper Montclair, NJ 07043

hubey@pegasus.montclair.edu
http://www.csam.montclair.edu/~hubey

Multiple Valued Discrete and Continuous (Fuzzy) Logics

H.M. Hubey, Associate Professor

Abstract

A whole host of various multiple valued and infinite-valued (fuzzy) logics are proposed, and their relationships to the well-known multiple-valued logics of Bochvar, Kleene, Lukasiewicz, and Godel, and also to the fuzzy logics are shown. The standard max(), and min() values used for OR, and AND in Zadeh logics are given in terms of the elementary functions. Furthermore, the liar paradox is discussed in terms of the fuzzy logics, and an infinite set of idempotent and continuous fuzzy logics are derived.

1. A Multivalued Logic: Logic of Contradiction

Consider the system defined by the set of operations where the $\equiv$ has been used in definitions;

❶	$\delta \equiv \phi' \otimes \beta'$	$\delta' \equiv \phi \oplus \beta$	
❷	$\phi \equiv \delta' \otimes \beta'$	$\phi' \equiv \delta \oplus \beta$	
❸	$\beta \equiv \delta \otimes \phi$	$\beta' \equiv \delta' \oplus \phi'$	
❹	$A \equiv \delta \oplus \beta \oplus \phi \oplus \delta' \oplus \beta' \oplus \phi'$		*Definition of variables.*
❺	$(A \otimes A \equiv A) \otimes (A \otimes A \equiv A)$		*Absorption properties*
❻	$(A \oplus B) \equiv (B \oplus A) \otimes (A \otimes B) \equiv (B \otimes A)$		*Commutativity*
❼	$(A \oplus (B \oplus C) \equiv (A \oplus B) \oplus C) \otimes (A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C)$		*Associativity*
❽	$(A \otimes B) \oplus (A \otimes C) \equiv (A \otimes (B \oplus C))$		*Distributivity*

The statements ❶-❸ define the primitive values instead of 0 and 1 or T and F, the operations $\otimes$, $\oplus$, and the complementation operator which is a prime (only for convenience). The operation $\otimes$ may be taken to be AND, but it's not clear yet if $\oplus$ is OR or XOR. The prime is used for complementation. Statement ❹ says that A is a variable and statement ❺ states that the absorption property holds for both operations. The definitions are highly recursive. Instead of providing the truth tables for the three-valued logics of Bochvar, Lukasiewicz and Kleene as given in Rescher [1969] or Gamut [1991], only the recursive definitions have been provided. The natural question to ask is if this system is defined (as in the previous sense i.e. defined consistently and recursively so as to allow the derivation of the tables for the operations $\oplus$ and $\otimes$). In the following calculations $A \otimes B$ will be written simply as AB. As before, the definition of complementation is a part of the relationships above. We can immediately derive some fundamental relationships that can be used to construct the addition and multiplication tables.

1) $\beta' \equiv \phi' \oplus \delta' = \phi \oplus \beta \oplus \delta \oplus \beta = \phi \oplus \delta \oplus \beta = \phi \oplus \phi' = \delta \oplus \delta' = \beta \oplus \beta'$

2) $\beta \equiv \delta\phi = \phi'\beta'\delta'\beta' = \phi'\delta'\beta' = \phi'\delta'(\phi' \oplus \delta') = \phi'\delta' \oplus \phi'\delta' = \phi'\delta'$

3) $\beta\beta' = \phi\delta(\phi' \oplus \delta') = \phi\phi'\delta \oplus \phi\delta\delta' = (\phi\delta)\phi' \oplus (\phi\delta)\delta' =$
$= (\phi'\delta')\phi' \oplus (\phi'\delta')\delta' = \phi'\delta' \oplus \phi'\delta' = \phi'\delta' = \beta$

We can summarize the most important relationships that are used constantly to derive the multiplication and addition tables.

4) $\beta = \beta\beta'$

5) $\beta' = \beta \oplus \beta'$

If we add β' to (9) we obtain $\beta \oplus \beta' = \beta' \oplus \beta\beta'$ which in view of (5) becomes

6) $\beta' = \beta' \oplus \beta\beta'$

If we multiply (6) by β we then obtain $\beta\beta' = \beta \oplus \beta\beta'$ which in view of (5) becomes

7) $$\beta = \beta \oplus \beta\beta'$$

From (4) through (7) we can see that the forms $\beta\beta'$ (i.e. β) and $\beta\oplus\beta'$ (i.e. β') behave like identities--β' is a multiplicative identity and β is an additive identity. It's not important for now that boolean algebras are lattices and that the words that are being used here such as identity might not be useful in that sense. If contrary to the general way in which it is viewed, we make zero (false) the multiplicative *attractor* (absorber), then $x \cdot 0 = 0$; and for addition we let unity(true) be the additive attractor we obtain $x + 1 = 1$. In other words the two new elements of this multivalued logic seem to behave just like the elements of standard boolean algebra on which SPL is based. The word attractor can be intutitively defined as follows: specifically, the primitive elements (values) of standard boolean algebra behave as;

8) $$FT = F$$

9) $$F + T = T$$

which are analogous to (4) and (5). Multiplying (14b) by T yields $FT + T = T$ and adding F to (9) yields $FT + F = F$ which are analogous to (6) and (7). However, we know from standard Boolean algebra that the *cancellation* (a simple word for both division or subtraction in some extended sense) is illegal in certain cases. For example, if we add T to (9) and without using the absorption property on the rhs but using it on the lhs we obtain

10) $$F + T = T + T$$

which yields (after *additive cancellation -i.e. addition*)

11) $$F + \cancel{T} = T + \cancel{T} \quad \text{i.e.} \quad F = T$$

and is clearly incorrect. Similarly multiplying (8) by F and then using the absorption property on the lhs but not on the rhs yields

12) $$FT = FF$$

If we now use the multiplicative cancellation (i.e. division) we derive

13) $$\cancel{F}T = \cancel{F}F \quad \text{i.e.} \quad F=T$$

clearly another incorrect result. There seems to be a direction in which operations can be performed which those who've habituated know is simple but difficult for beginners to grasp if they're only used to ordinary arithmetic. There's a very simple way in which the rules can be explained and that is in terms of an *attractor.*

The truth value F is a *multiplicative attractor* and T is an *additive attractor*, since Fx=F and x+T=T. It just so happens that in the case of bivalent logic, the multiplicative attractor is the additive inverse and the additive attractor is the multiplicative inverse. Thus we write F+x=x (or 0+x=x) and Tx=x (or $1 \cdot x = x$) instead of using the idea of an *attractor*. We could have just as easily expressed these as; T+x = T and $F \cdot x = F$. The two systems are isomorphic. But the idea of an attractor makes the preceding seemingly complicated one-way operations with Boolean valued variables much simpler. A more thorough discussion of these ideas is given later [Chapter VI,Hubey,1996]. In the case of the multivalued system whose definitions of its primitive elements, and operations are given in recursive fashion as above, all we have to do now is to compute the products and sums. After this we can examine the tables to see what kind of a system the definitions produce. Before deriving all the elements of the table it should be noted that;

14) $\delta'\beta' = \delta'(\phi'\oplus\delta') = \delta'\phi'\oplus\delta' = \beta\oplus\delta' = \beta\oplus\phi\oplus\beta = \phi\oplus\beta = \delta' = \phi$

15) $\phi'\beta' = \phi'(\phi'\oplus\delta') = \phi'\oplus\phi'\delta' = \beta\oplus\phi' = \beta\oplus\delta\oplus\beta = \beta\oplus\delta = \phi' = \delta$

Since $\delta'=\phi$ and $\phi'=\delta$, there's no need to include the complements δ' and ϕ' in the tables. It's obvious now that if we assign the values δ=T, ϕ=F, then β becomes interpreted as *both true and false*. We can see that T (δ) and F (ϕ) are complements. Furthermore, β and β' also have the same relationships to one another as T and F as given in the table below.

$\otimes$

	T	F	B	B´
T	T	B	B	T
F	B	F	B	F
B	B	B	B	B
B´	T	F	B	B´

$\oplus$

	T	F	B	B´
T	T	B´	T	B´
F	B´	F	F	B´
B	T	F	B	B´
B´	B´	B´	B´	B´

It can clearly be seen that in multiplication, the B´ is the attractee (identity) and B is the attractor. Moreover, in addition, as we saw for standard Boolean algebra, B (the complement of B´) becomes the attractee and B´ becomes the attractor . Using the semantics that we assigned we can rewrite the definitions and show their ramifications and connotations

16a) $T=F'B'=TB'$

16b) $T'=F=F\oplus B=F+TF=F+F=F$

16c) $F=T'B'=FB'$

16d) $F'=T=T\oplus B=T+TF=T+F=T$

16e) $B=TF=TB'FB'=TFB'=BB'=B$

16f) $B'=T'\oplus F'=T\oplus F=TB'+FB'=B'(T+F)=B'B'=B'$

We can show further ramifications of these definitions, for example

17a) $T = TB=T(T\oplus F)=T\oplus TF=T\oplus B$

17b) $B'=T\oplus F=TB\oplus FB=B(T\oplus F)=BB'$

One disadvantage of this system is that we can't just drop the B and B´ and still be left with the classical system. And the B and B´ also show up in the sums and products of the classical values of T and F, which doesn't occur in Kleene, Bochvar and Lukasiewicz systems. However, this system will reduce to the classical system if we simultaneously assign *True* to B′ and *False* to B. In a sense this logic is multiplicative rather than additive. In another sense, it is still the same classical logic in that the contradiction has been assigned a truth value of its own. A big disadvantage, of course, is that the system is not truth-functional and hence is not logic in the standard sense. For example modus ponens computes to B′ hence is not a tautology. However, if we simultaneously interchange T with B′ and F with B, everywhere, the system [addition and multiplication] is isomorphic to Lukasiewicz four-valued product system and modus ponens becomes a tautology. The trivalent systems are forced to make the third truth value self-complementing so that it's difficult to give it any meaning other than unknown or indeterminate. If we attempt to produce such values via extrapolation from the bivalent case, it requires a reinterpretation of OR since T+F can be interpreted as True or False but unkown as to which is the value. In this sense, the system above can be made to collapse to various ternary systems. For example, if we assign B to *False* and retain B′ as an indeterminate value then addition becomes isomorphic to the Bochvar system, and implication computed as P′+Q is also the same as in the Bochvar system.

2. Contradiction of Logic of Contradiction

Definitions of primitives and their complements and the operators via generalized complementation are given below for an alternative four-valued logic.

❶ $\delta \equiv \eta' \otimes \phi'$ $\delta' \equiv \eta \oplus \phi$

❷ $\phi \equiv \eta' \otimes \delta'$ $\phi' \equiv \delta \oplus \eta$

❸ $\eta \equiv \phi' \otimes \delta'$ $\eta' \equiv \phi \oplus \delta$

❹ $A \equiv \delta \oplus \eta \oplus \phi \oplus \delta' \oplus \eta' \oplus \phi'$ *Definition of variables*

❺ $(A \otimes A \equiv A) \otimes (A \otimes A \equiv A)$ *Absorption properties*

❻ $(A \oplus B) \equiv (B \oplus A) \otimes (A \otimes B) \equiv (B \otimes A)$ *Commutativity*

❼ $(A \oplus (B \oplus C) \equiv (A \oplus B) \oplus C) \otimes (A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C)$ *Associativity*

❽ $(A \otimes B) \oplus (A \otimes C) \equiv (A \otimes (B \oplus C))$ *Distributivity*

The motivation for this system can be seen in the definition of η. In the previous system we had defined β as *Both* (True and False). Accordingly we might give the meaning *Neither* (neither true nor false) to η. This is not exactly correct since it seems that *Neither* should really be defined as NOT(T or F). However, taking the cue from the previous system that the Law of the Excluded Middle does not hold so that T + F is not T; that T and F might not be negations of one another in this system and furthermore by an intuitive extension of the semantics of De Morgan's Laws so that NOT(T or F) might be representable as $T'F' = \delta'\phi'$. We may also think of *Neither* as $T' + F'$ which if we try to to extrapolate via De Morgan's Laws should be equivalent to $(TF)' = T' + F' = T+F$ or $(T+F)' = T'F' = FT$.

So it does seem as if *Neither* is and is not really *Both*. We can immediately show

18) $\delta' = (\eta'\phi')' = \phi \oplus \eta$ and hence $(\delta')' = \eta'\phi' = (\phi \oplus \eta)'$

19) $\phi' = (\delta'\eta')' = \delta \oplus \eta$ and hence $(\phi')' = \eta'\delta' = (\delta \oplus \eta)'$

20) $\eta' = (\phi'\delta')' = \phi \oplus \delta$ and hence $(\eta')' = \phi'\delta' = (\phi \oplus \delta)'$

These can be written in general by using variables to derive the generalized complements

21) $(A\oplus B)' = A'B'$ and $(A'B')' = A\oplus B$

It's not clear that substitution of complements will work since we have three values in which complementation must take place. Meanwhile we can compute some other relationships that exist among the operators and the primitive values(where for convenience we'll be writing AB for $A\otimes B$)

22a) $\delta\phi = (\eta'\phi')(\eta'\delta') = \eta'\phi'\delta' = \eta'\eta$
22b) $\delta\eta = (\phi'\eta')(\delta'\phi') = \phi'\delta'\eta' = \eta\eta'$
22c) $\phi\eta = (\delta'\eta')(\delta'\phi') = \delta'\phi'\eta' = \eta\eta'$
22d) $\eta\eta' = \delta'\phi'\eta' = \delta\delta' = \phi\phi'$

We derive some more fundamental relationships which will be of use in the construction of the addition and multiplication tables.

23) $\delta'\oplus\phi' = \phi\oplus\eta\oplus\delta\oplus\eta = \phi\oplus\delta\oplus\eta = \eta\oplus\eta'$
24) $\eta = \delta'\phi' = (\phi\oplus\eta)(\delta\oplus\eta) = \delta\phi\oplus\phi\eta\oplus\delta\eta\oplus\eta = \eta\oplus\eta\eta'$
25) $\eta' = \delta\oplus\phi = \phi'\eta'\oplus\delta'\eta' = \eta'(\phi'\oplus\delta') = \eta'(\eta\oplus\eta') = \eta'\oplus\eta\eta'$

These are of the same form as equations (13) and (14). For example, adding (24) to (25) we derive

26a) $\eta\oplus\eta' = \eta\oplus\eta'\oplus\eta\eta'$

We could have also derived it via the definitions as below

26b) $\eta\oplus\eta' = \delta'\phi'\oplus\delta\oplus\phi = (\phi\oplus\eta)(\delta\oplus\eta)\oplus\delta\oplus\phi =$
$= (\delta\phi\oplus\phi\eta\oplus\delta\eta)\oplus\eta(\oplus\delta\oplus\phi) = \eta\eta'\oplus(\eta\oplus\eta')$

If we multiply (25) by η we obtain

27) $\eta\eta' = \eta(\eta'\oplus\eta\eta') = \eta\eta'\oplus\eta\eta'$

If, as before we, assign these values, we'll be able to generate a consistent system

28) $\eta = \eta\,\eta'$ and $\eta' = \eta\oplus\eta'$

If we accept these identities, the relationships (24) through (27) are also consistent. We can see some of the ramifications of this, for example

29a) $\delta\oplus\eta\eta' = \delta\oplus\eta = \phi'$ and similarly $\phi = \delta'$
29b) $\delta(\eta\oplus\eta') = \phi'\eta'(\eta\oplus\eta') = \phi'(\eta\oplus\eta') = \phi'$

The full multiplication and addition tables are given further below. The original definition of δ was (equation (i) above), then, not *False* and not *Neither*. hence it might be construed as denoting decidable/knowable truth.

For some types of reasoning, for example those employing a future contingent such as the sentence "There will be a sea battle tomorrow", intuitively we want to say that it's *Neither* (true nor false) but not *Both* (true and false). The easy way out is to declare the sentence to be a meaningless statement since assigning either a true or a false value to it implies that we are able to tell the future. In this case, the semantics of meaningless may correspond more closely to *Neither* than *Both*. In the case in which we avoid giving it a truth value, it seems to also imply that it must have yet another truth value. Despite strenuous objections to the contrary it seems that truth value gaps are also values themselves. When we say that something is at rest we can construe it to mean that it has no velocity but in fact it's even easier and more correct to say that its velocity is zero. In order to claim that something at rest has no velocity value is equivalent to the assertion that zero is not a number and hence not a number value. Attempts to solve paradoxes by sticking to bivalent logic and then assigning no values to certain propositions or statements seem to be producing and not producing more truth values.

If we think of the statement "Jane is not tall and not short" as saying that Jane is of medium height then it does seem as if *Neither* can be equivalent in some sense to *Both*. For if *tall* and *short* are complements then the expression "not tall and not short" is essentially $T'S'$ which in view of the supposed complementarity of T and S is TT' which can easily be interpreted as *Both* (tall and not tall). Similarly if we think of *tall* and *short* as ranges of height and if $T'S'$ means *medium stature*, then this medium stature must overlap both of the intervals for tall and short (where tallness and shortness is continous valued) and hence is in effect Both. In order to be able to make this clearer we have to make appeals to modulo arithmetic or even infinite valued arithmetic or logic.

In ordinary speech, *Both* is probably reserved for situations in which a generalization has components that are true and some that are false. Thus a sharper and clearer statement might yield a True or False answer, and thus *Both* might be similar in meaning to paradoxical. However, *Neither* seems more like meaningless or undecidable or unknown; in the sentence above, there can be no sharper or unambiguous restatement that will result in anyone being able to assign a True or a False value to that statement.

But if we want to include *Neither* as a possibility we should then probably also include *Both* as a possibility; in neither of these logics have we tried to include both *Neither* and *Both* as legal values. Hence in this case too, the Law of the Excluded middle is avoided and the case of both True and False is not automatically assigned a value of False although Both is not allowed, at least in the definition. However we can easily compute the value of both True and False.

30) $$\delta\phi = \phi'\eta'\delta'\eta' = \phi'\delta'\eta' = \eta\eta' = \eta = \eta' = \eta\oplus\eta'$$

This paradoxical result could be due to the fact that our definitions did not allow for Both which would have been defined as $\beta = \delta\phi$. In any case, in order to restrict this system to produce a kind of a system which we intuitively accept to be correct, we have to introduce some extra constraints into the system than the ones we've introduced in the definitions. It seems that if we allow the definition of Neither, it inevitably becomes equivalent to Both. In the beginning we could have defined *Neither* as $\eta \equiv (\delta\oplus\phi)'$ and $\eta' \equiv \delta\oplus\phi$. Then we'd have

31) $\eta = (\phi'\eta'\oplus\delta'\eta')' = (\eta'(\delta'\oplus\phi'))' = (\eta'(\phi\oplus\eta\oplus\delta\oplus\eta))' =$
$=(\eta'(\delta\oplus\phi\oplus\eta))' = (\eta'(\eta \oplus \eta'))'$

Therefore we'd conclude

32) $$\eta' = \eta'(\eta\oplus\eta') = \eta\eta'\oplus \eta'$$

We can produce the multiplication and addition tables from these relationships. As can be seen the tables are isomorphic to the tables for the multivalued logic which includes B and B´. N becomes the attractor for ⊗ and N´ the attractee (identity). For ⊕, the roles become reversed. The fact that we get the nonstandard values of N and N´ for the interactions of T and F are disconcerting, however this system also collapses to the standard system if we let N→F and N´→T. The definitions are given once more using T,F, N and N´.

33a)	$T=F'N'=F'=TN'=T$	$T'=F\oplus N=F$
33b)	$F=T'N'=T'=FN'=F$	$F'=T\oplus N=T$
33c)	$N=T'F'=TF$	$N'=T\oplus F=T'\oplus F'$

The definitions in (33c) are also consistent with other derivations; for example;

34a) $N=T'F'=(F\oplus N)(T\oplus N)=FT\oplus FN\oplus TN\oplus N=F'N'T'\oplus T'N'T'F'\oplus F'N'T'F'\oplus N=$
$=NN'\oplus NN'\oplus NN'\oplus N=NN'\oplus N=N\oplus N=N$

34b) $N'=T\oplus F=F'N'\oplus T'N'=N'(F'\oplus T')=N'(F\oplus N\oplus T\oplus N)=$
$=N'(T\oplus F\oplus N)=N'(N'\oplus N)=N'\oplus NN'$

⊗

	T	F	N	N´
T	T	N	N	T
F	N	F	N	F
N	N	N	N	N
N´	T	F	N	N´

⊕

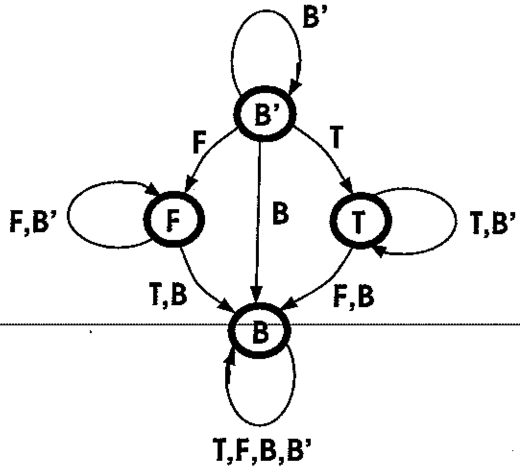
	T	F	N	N´
T	T	N´	T	N´
F	N´	F	F	N´
N	T	F	N	N´
N´	N´	N´	N´	N´

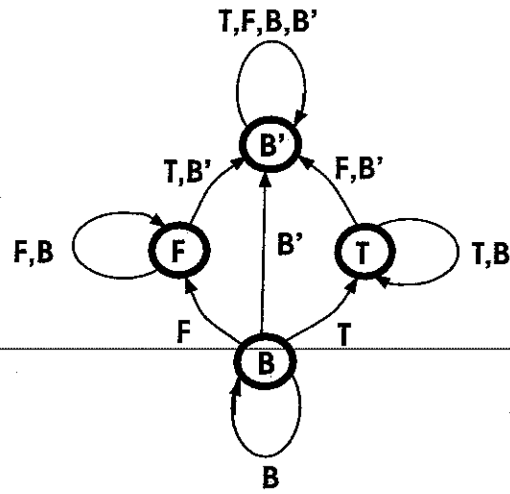
We could have tried a different set of definitions such as

(35)	$\delta \equiv (\phi\oplus\eta)'$	$\delta' \equiv \phi\oplus\eta$	or	$\delta' \equiv \phi'\eta'$
	$\phi \equiv (\delta\oplus\eta)'$	$\phi' \equiv \delta\oplus\eta$	or	$\phi' \equiv \delta'\eta'$
	$\eta \equiv (\delta\oplus\phi)'$	$\eta' \equiv \delta\oplus\phi$	or	$\eta' \equiv \delta'\phi'$

or even

(36)	$\delta \equiv (\phi\oplus\beta)'$	$\delta' \equiv \phi\oplus\beta$	or	$\delta' \equiv \phi'\beta'$
	$\phi \equiv (\delta\oplus\beta)'$	$\phi' \equiv \delta\oplus\beta$	or	$\phi' \equiv \delta'\beta'$
	$\beta \equiv (\delta\oplus\phi)'$	$\beta' \equiv \delta\oplus\phi$	or	$\beta' \equiv \delta'\phi'$





3. Bang-Bang Infinite Valued Logic

Once we start to make logical variables functions of time (or in reality functions of any number of parameters), then all the apparatus of related disciplines becomes available for exploitation. Continuous valued logic such as fuzzy logic is capable of making use of much of the machinery of analysis and as a result has settled somewhere in the transition zone between bivalent logic, probability theory and discrete-multivalent logic. It's easy to define the conditional in terms of some of the commonly used functions of analysis. For example, it is easily seen that the implication $A \Rightarrow B$ really implements in two values the relationship $t(A) \leq t(B)$; that is, the conditional is false if the truth value assigned to A is not greater than or equal to that of B. It's common in discrete multivalent logics and infinite valued logics to define the truth-value of a variable as that of itself; that is $t(A)=A$ and $NOT(t(A))=1-A$. To produce crisp logic from these valuations then, the simplest function to use is a threshold type of function, say the Heaviside Unit Step Function, $U(A)$. Instead of sets, because of the isomorphism between set theory, Boolean Algebra and propositional logic, the following will concentrate only on the truth-value aspects or the propositional aspects of the related theories, instead of the usual concentration on sets and membership functions. If instead of defining the truth value of a continuous valued variable to be the value of the variable itself [restricted to the interval [0,1]], we allow the variable to range over some other interval [0,L] and then define its truth value by another function, then we can produce a logic isomorphic to standard bivalent logic at the boundaries quite easily. We can simply define the truth value of a [propositional/Boolean] variable $0 \leq A \leq 1$ to be crispified by the Heaviside Unit Step Function

37) $$t(A) = U(A-\beta)$$

where $0 < \beta < 1$ is the bias, which for practical purposes should be 1/2 and where $U(x)$ is the unit step function which is defined to be 1 for $x \geq 0$ and 0 for $x < 0$. In some books $U(0)$ is defined as 1/2. Then the complement of the variable is quite easily defined as

38) $$NOT(A)=U(\beta-A)$$

Functions such as $U(x)$ could be called *quasi-odd* in that if instead of $U(x)$ we had used, say, the sign function, signum(x) we would get toggling of the values +1,−1,+1,−1.. as the value of the argument is negated. In the case of the quasi-odd functions it toggles between 0 and 1 if the argument of the function is negated, that is if we substitute 1−A, which is the common negation of infinite valued logical variables.

In the symmetric case when $\beta = 1/2$, negation or complementation, substituting 1−A into U(A−1/2) produces U(1/2−A) which as shown above is the complement. The subleties required for certain logics, for example distinguishing between negation and complementation are not required here. Instead of simply producing a negation via a threshold we've defined it via its truth-valuation. We note that it is desirable that the complement function satisfy these properties [see for example Klir & Folger]:

C1. NOT(0)=1 and NOT(1)= 0	*Boundary Conditions i.e. crisp logic*
C2. $\forall a,b \in [0,1]$ [if A< B then NOT(A)$\geq$ NOT(B)]	*Monotonic nonincreasing*
C3. NOT(A)	*[is] continuous*
C.4 NOT(NOT(A))= A for all A$\in$[0,1]:	*Involutive*

It should be noted that U(β−x) satisfies C.1, and C.2. In addition it satisfies a condition of which C.4 is a special case. The unit step function definition satisfies

C.5 NOT(NOT(t(A))= t(A) where NOT(t(A))= t(1−A)
NOT(NOT(U(A−1/2))=NOT(U(1/2−A)=U(A−1/2)

In the case in which we set t(A)=A, then this condition is C4. Then it's an easy matter to show that the OR and the AND functions can be defined as

39a) $\mathrm{OR}(A,B) \equiv U(A+B-\beta)$

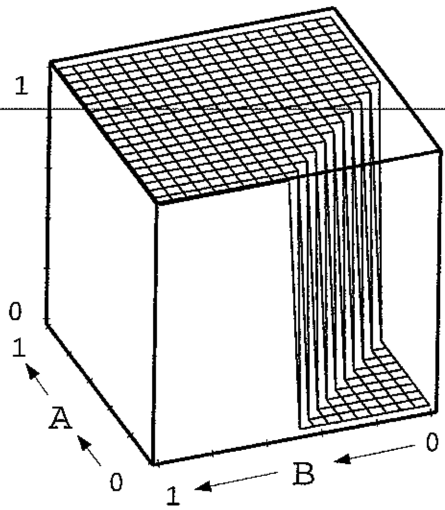
39b) $\mathrm{AND}(A,B) \equiv U(AB-\beta)$

where the operations in the argument of U(x) are ordinary arithmetic operations such as addition, subtraction and multiplication. It would be possible to define AND first and then define the others in terms of NOT and AND as will be shown for other infinite valued logic systems in later sections. Some desirable properties that the OR should possess are:

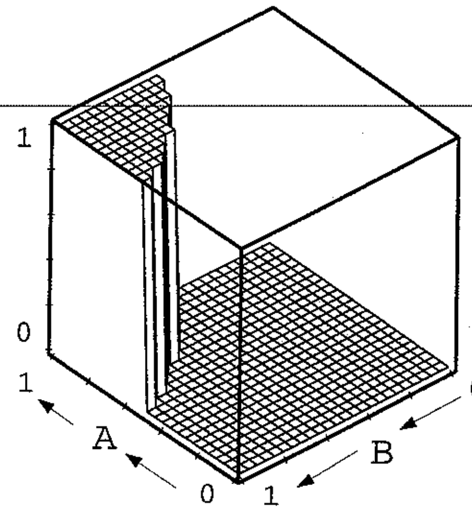
O.1	OR(A,B) should satisfy the B.C.	
O.2	OR(A,B) = OR(B,A)	*commutative*
O.4	OR(A,B) $\leq$ OR(A′,B′) for A$\leq$ A′ and B$\leq$ B′	*monotonic*
O.4	OR(OR(A,B),C) = OR(A,OR(B,C))	*associative*
O.5	OR(A,B) should be continuous	
O.6	OR(A,A) = A	*idempotent*

It can be verified that the U(A+B−1/2) meets all the requirements except for O.5 and O.6. If however, we consider t(A), that is U(A−1/2) instead of A, then it may be argued that the idempotency condition is satisfied. Similar comments apply regarding the definition of AND as U(AB−1/2). The conditions for the AND(A,B) [or intersection] are similar to the OR(A,B).

A.1	AND(A,B) should satisfy the B.C.	
A.2	$AND(A,B) = AND(B,A)$	*commutative*
A.4	$AND(A,B) \leq AND(A',B')$ for $A \leq A'$ and $B \leq B'$	*monotonic*
A.4	$AND(AND(A,B),C) = AND(A,AND(B,C))$	*associative*
A.5	AND(A,B) should be continuous	
A.6	$AND(A,A)= A$	*idempotent*



OR(A,B) = U(A+B−1/2)



AND(A,B) = U(AB−1/2)

Having defined OR, AND and NOT we can show that the conditional can be written in terms of these as;

40a) $(A \Rightarrow B)_1 = \neg(\neg BA) = U[\beta - U[A - U(\beta - B) - \beta]]$

40b) $(A \Rightarrow B)_2 = \neg A + B = U[B + U(\beta - A) - \beta]$

It can be verified via substitution in the manner of constructing truth tables that the NAND and NOR functions are related to the OR and the AND via the negation introduced earlier so that we have [where β=1/2 for symmetry].

41a) $NAND(A,B) \equiv \neg(A+B) \equiv U(\beta - AB)$

41b) $NOR(A,B) \equiv \neg(AB) \equiv U(\beta - (A+B))$

41c) $XOR(A,B) \equiv U\{U[B \cdot U(\beta - A) - \beta] + U[A \cdot U(\beta - B) - \beta] - \beta\}$

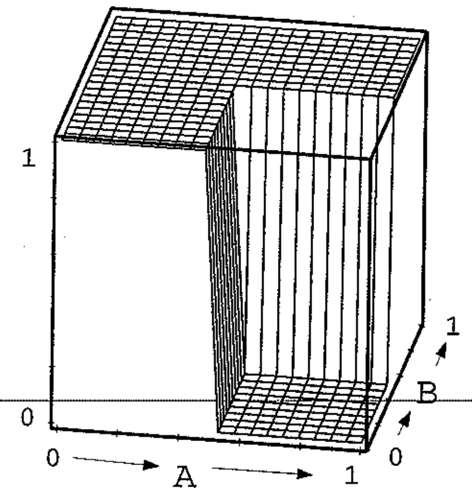
41d) $EQ(A,B) \equiv U\{U[A \cdot U(\beta - A) - \beta] + U[B \cdot U(\beta - B) - \beta] - \beta\}$

Using the definition of negation we obtain the identities

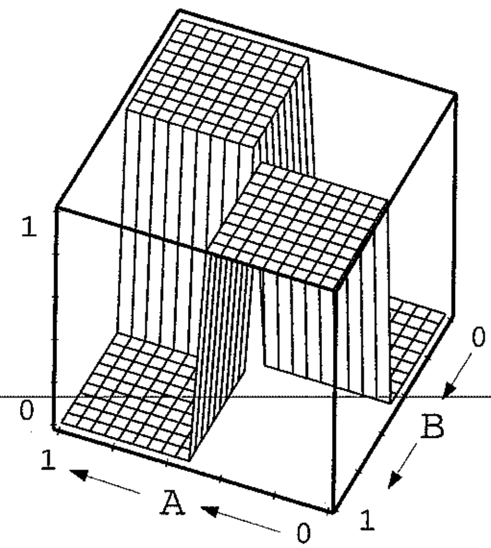
42a) $A+B = U(\underline{A+B} - \beta) = U[\beta - U(\beta - A)U(\beta - B)]$

42b) $A \Rightarrow B = \neg A + B = U(B - A) = \neg(\neg BA) = U[\beta - U[A \cdot U(\beta - B) - \beta]]$

42c) $\neg(A+B) = U[\beta - (A+B)] = (\neg A)(\neg B) =$



$A \Rightarrow B = U[B + U(1/2 - A) - 1/2]$
$= U[1/2 - U(A \cdot U(1/2 - B) - 1/2)]$



XOR(A,B)

$$= U[(\neg A)\cdot(\neg B) - \beta] = U[U(\beta - A)\cdot U(\beta - B) - \beta]$$

42d) $A+B = B+A = U(\neg A - B) = U(\neg B - A) =$
$$= U[U(\beta - A) - B] = U[U(\beta - B) - A]$$

42e) $U(\beta - AB) = U[\beta - U(AB - \beta)] = U(\neg B - A) = U(\neg A - B)$

42f) $U\{U[B\cdot U(\beta - A) - \beta] + U[A\cdot U(\beta - B) - \beta] - \beta\} =$
$$= U\{\beta - U[A + U(\beta - B) - \beta]\cdot U[B + U(\beta - A) - \beta]\}$$

42g) $U\{U[A\cdot U(\beta - A) - \beta] + U[B\cdot U(\beta - B) - \beta] - \beta\} =$
$$= U\{\beta - U[U[U(\beta - A) + U(\beta - B) - \beta]\cdot U[A + B - \beta] - \beta]\}$$

One of the benefits or problems, depending on one's point of view, is that any arbitrary function of the parameters A, and B as defined with the step function always yields one of the standard Boolean functions at the boundaries. The only time a value is not 0 or 1, is during the initial assignment, so that this is really classical logic or at least isomorphic to it at the boundaries so that it may be considered to be an arithmetization of propositional calculus. It should be noted that the implication is treated as the infinite valued analogue of the standard bivalent implication, that is the Kleene-Dienes implication and not the Mamdani implication that is often used in control application. In the following sections, there will be interpretation of these infinite-valued logics as densities, binary discrimination/perception functions or as control/effect functions.

4. Knife-Edge Bang-Bang Infinite Valued Logic

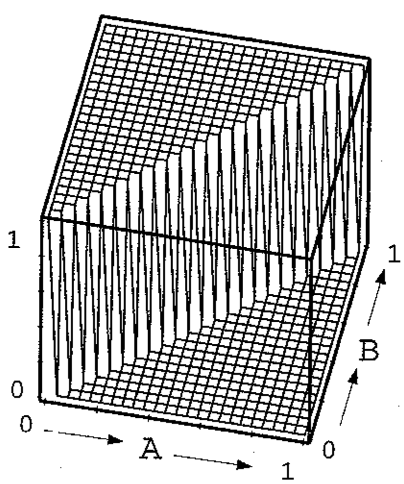
We can easily create other logics with different values which will be isomorphic to classical bivalent logic at the boundaries.Taking a hint from the earlier interpretation that the conditional makes a statement that *$t(A) \leq t(B)$ is true*, then it's easy to see that the conditional can be written as

43) $(A \Rightarrow B) \equiv U(B - A)$ and $(B \Rightarrow A) \equiv U(A - B)$

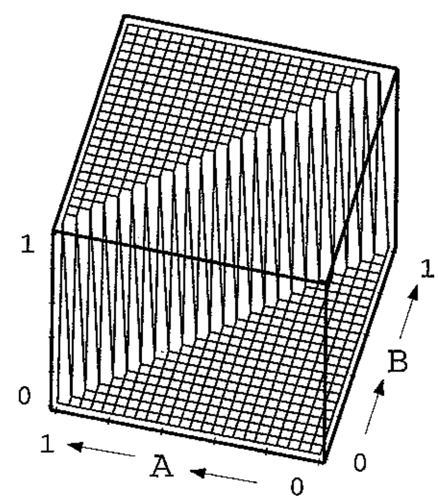
The highly nonlinear and discontinuous nature of the step function make it obvious that arithmetization of logic is accomplished via highly nonlinear and iterative/recursive functions. Its relationship to probability theory, fuzzy logic or Carnap's probabilistic logic cannot be a very simple one. Since we have already defined negation then all the other functions can be derived from negation and implication. In terms of some of the earlier developed functions, the seemingly innocuous conditional if *fully algebraized* can be written more simply as (Hubey[1997])

$$A \Rightarrow B \equiv \frac{1}{2}+\frac{B-A}{2\sqrt{(B-A)^2}}$$

since for B >A the second term is +1/2 and for B < A it is –1/2. It is, in fact, zero for A > B. The knife-edge problem occurs at A= B. The limit from the right (that is from B>A) is unity and the limit from the A< B



(A⇒B)=U(B–A)
Knife-Edge Implication



OR(A,B)=U(B–U(B-A))
Defined Via Implication

direction is zero. Since the unit step function is sometimes defined as 1/2 at zero for this reason , the equation above will work as a definition of implication with the convention that the value of the equation above is unity for A=B. This is also one of the definitions of the Heaviside step function [for example the CAS Maple uses this definition]. It is convenient in this knife-edge logic to define the OR as done by Lukasiewicz for multivalent logic

45) OR(A,B)≡ (A⇒B)⇒B ≡ U[B–U(B–A)]

Then we can define AND(A,B), and NAND(A,B) via De Morgan's Laws, and then the XOR(A,B) using AND and OR.

46a) NAND(A,B)≡ U{U(1/2–B)–U[U(1/2–B)–U(1/2–A)}

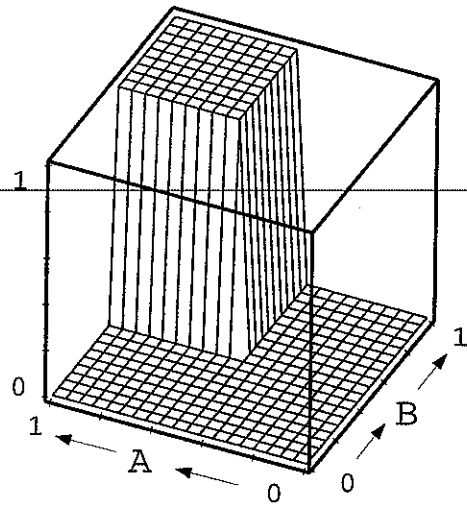
46b) AND(A,B) ≡ U[1/2–U{U(1/2–B)–U[U(1/2–B)–U(1/2–A)}]

46c) XOR(A,B)≡ U{AND(B,U(1/2-A)) – U[AND(B,U(1/2–A))–AND(A,U(1/2–B))]}

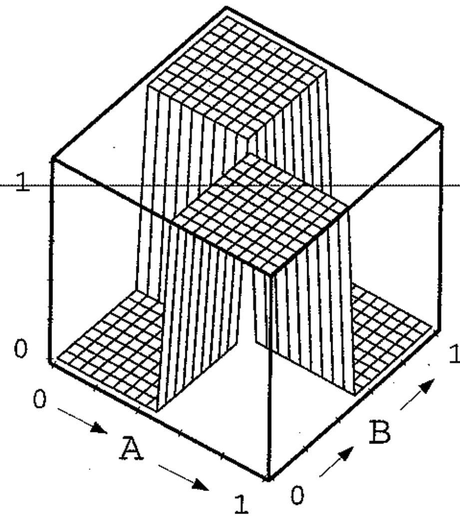
The NAND(A,B) is directly from the definition of OR via De Morgan's Laws and the AND(A,B) is via a negation of the NAND(A,B). Anything directly connected with implication such as modus ponens and modus tollens can be expressed more simply as

47a) $$(A(A\Rightarrow B)\Rightarrow B) \equiv U[B - A\cdot U(B - A)]$$

47b) $$(\neg B(A\Rightarrow B)\Rightarrow \neg A) \equiv U[U(1/2-A) - U(1/2-B)\cdot U(B - A)]$$



Knife-Edge AND(A,B)



Knife-Edge XOR(A,B)

In contrast to attempts to simplify or linearize logic (or in fact everything!) we seem to have discovered nonlinear iteration in logic. Even piecewise linear equations are not linear, and neither is abs(x). The step functions are even more complex than abs(x) functions or piecewise linear functions. The strange thing is that despite the seeming knife-edge stability of the implication as defined earlier in this section, both the modus ponens and the modus tollens obtain the value unity throughout the region in which A and B are defined. Because of the equivalence of the contrapositive we have

47c) $$U(B-A)= U[U(1/2-A)- U(1/2-B)]$$

Other relationships can be derived from these. The explicit assigment of truth values to logical variables, that is, the separation of the variable value and its truth value via the introduction of another level of indirection provides a more clear cut discussion of the Liar Paradox. The Liar Paradox, in terms of the unit step function valuation can be simply expressed as one of;

48a) $$t(P)=U(P-1/2) :=U(1/2-P) \qquad [\text{or } P :=U(1/2-P)]$$

48b) $$t(P_{n+1})=U(P_{n+1}):= U(1/2-P_n)$$

where the assignment statement has been made explicit since it is the act of attempting to assign a truth value that causes the problems with impredicative paradoxes of this type. Casting it as in (48b) also explicitly shows the iterative nature of the paradox. The Truth-teller Paradox is simply

48c) $$t(Q) := U(Q-1/2)$$

48d) $$t(Q_{n+1}) := U(Q_n-1/2)$$

It should be noted that both views, i.e. that of P representing the variable value and also the truth value and also separating these into two distinct ideas, are used simultaneously in the above equations hoping it will not lead to confusion. The value for the bias is 1/2 as elsewhere to avoid the special problems and complexity that arise from not having an involutive complementation. However, if there's no bias in the step function, then we can always produce another paradox exactly at the value of 1/2.

5. Invariants of Logic

The apparently infinite-valued logics of the previous sections really were confined only to the boundaries because of the properties of the unit step function. However, the unit step function U(x) is a good model for truly continuous valued logics since we need to examine other functions which like U(x) are 0 and 1 at the boundaries and smooth. Because of the continuous nature of these logics(!) instead of the upper case letters as for the propositional variables, lower case letters will be used. If we want to take some guesses as to what kinds of laws of logic are impeccably true and should be preserved, the three that are commonly put forward as candidates are:

51a)	$x+C(x)=1$	*The Law of the Excluded Middle (LEM)*
51b)	$C(x\cdot C(x)) = 1$	*The Law of Non-contradiction (LNC)*
51c)	$C(C(x)) = x$	*TheLaw of Involution or Self-Inverse (LSI)*

where C(x) is the negation or complement function which serves to repudiate, revoke, recant, revoke, retract, void, undo x. LNC is usually written as $x\cdot C(x)= 0$ and called the Law of Contradiction. It can be verified that if instead of using the logical variables themselves we use the truth values assigned to the logical variables, t(x) in the equations above, then the simplest case of t(x)=x and NOT(t(x))= NOT(x) = 1−x satisfy (51a), and (51c). However then this particular truth value assignment, t(x) then makes $x\cdot C(x)$ compute to $x(1-x)$ which is definitely not zero anywhere except at the boundaries. Adding another level of indirection to continuous valued logic by separating of the truth value assigned to the logical variable from the value of the logical variable allows the satisfaction of some of the constraints above by functions other than t(x) = x. For example if the variable was allowed to vary in the interval [0,L], we can define t(x)= x/L and NOT(t(x)= L−x/L = (L−x)/L. Using this idea, it can then be seen that $t(x)=U(x-\beta)$ [where x is constrained to [0,1] here for simplicity of the presentation] also satisfies the conditions of (51).

52a)	$U(x-1/2) + U(1/2-x) = 1$
52b)	$U[1/2-U(x-1/2)\cdot U(x-1/2)]=1$
52c)	$U[1/2-U(1/2-x)] = x$

In the light of the behavior of the step function we might derive a different kind of logic by defining $1/2 \le t(x) \le 1$ as truth and $0 \le t(x) < 1/2$ as falsity. The value of 1/2 has been assigned to truth for the simple reason that U(0)=1 and is not important from a philosophical view. [In some places one finds U(0)=1/2.] In this particular case, the unit step function not only serves to assign a truth value t(x) to the variable x but also serves to crispify it since it only takes on the classical logic's values of zero and one. It should be noted that the step function also satisfies

53a)	$NOT(x)= NOT(U(x-1/2))=U((1-x)-1/2)=U(1/2-x)$
53b)	$NOT(NOT(x))= U(1/2-(1-x))=U(x-1/2)$

where NOT(x) could really be written as t(NOT(x)) and f(x) is a truth value assigned to the variable x. It is not necessary for f(x) to have an inverse. Compared to the behavior of other functions when iterated these functions are not only not chaotic but they do not display any kind of period/frequency doubling nor any change in their successive iterated values. It might be appropriate to call such functions *ultra-stable* since the word super-stable has already been expropriated. It is seen that (53) satisfies a more general type of complementation or negation since instead of $t(x)=x$ and $t(NOT(x))=1-x$ we use $t(x)=U(x-1/2)$ and $NOT(x)= U((1-x)-1/2) = U(1/2-x)$. The floor function also satisfies (54) and can be used to produce multivalent logics. It should be noted that the step function can be used to crispify any infinite or multi-valent logic.

6. Some Useful Functions

A simple function such as abs(x) can be written, without using any kind of piecewise definition [which implicitly uses logical connectives] as;

54) $$\text{abs}(x) \equiv x^2/(+\sqrt{x^2})$$

It's a simple matter to show that the derivative of this function is the signum function;

55a) $$\text{signum}(x) \equiv 2x/(x^2)^{1/2} - x^3/(x^2)^{3/2} = 2x(x^2)^{-1/2} - x^3(x^2)^{-3/2}$$

The right limit [limit from the right] of signum(x) as defined above is 1 and the left limit is -1, exactly as it should be. Of course the derivative of the signum function can't be anything other than the Dirac delta function, for which we can provide a closed form expression

55b) $$\delta(x)= 2(x^2)^{-1/2} - 5x^2(x^2)^{-3/2} + 3x^4/(x^2)^{-5/2}$$

However, it can be verified that the signum function can be defined more simply

55c) $$\text{signum}(x) \equiv x/\sqrt{x^2}$$

The denominator is always positive and equal in magnitude to the numerator hence the abs(x) function may be considered to be derived from this function. Once we have these functions, then it's as simple to define the Unit Step Function

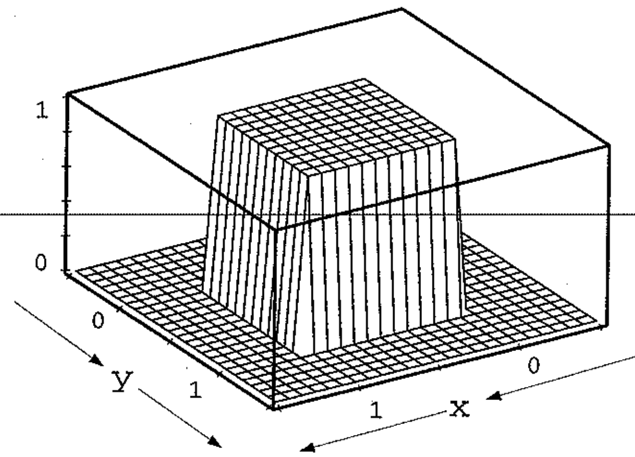
56) $$U(x) \equiv [1+ \text{signum}(x)]/2$$

It can be verified that the limit from the right of U(x) is unity and the limit from the left is zero. To be able to define (in the same sense as above i.e. from elementary functions) a *standard pulse function* Q(x), so that we can produce bona fide truth functions from elementary arithmetic functions instead of using piecewise definitions (which make implicit use of logical operators) so as to avoid having recursion and infinite regress in the definitions, we can use the U(x) as defined previously. Then

57) $$Q(x) \equiv U(x)U(1-x) = U(x) - U(1-x)$$

It is obvious that this pulse Q(x) has the value unity in the interval [0,1] and is zero elsewhere. However, it's possible to define Q(x) directly via a parabolic equation of form

58) $Q(x)= q(x;c,d) = 1/2 + [d^2/4 - (x-c)^2]/[2\{[d^2/4 - (x-c)^2]^2\}^{1/2}]$



Two Dimensional Parabolic Unit Pulse
Q(x)Q(y)

where d is the duration of the pulse centered at x = c. So the pulse equivalent to (57) is given by c = 1/2 and d=1. The limits of the unit [standard] pulse Q(x) as defined by either (57) or (58) are equivalent; the *outward limits* (i.e. limit from the left at x=0 and the limit from the right at x=1) are equal to 1, and the *inward limits* are 0. This last property is important when we define piecewise functions since we can then force such functions to take on the values at the intersections of the intervals belonging to that specific interval by using the convention of making the values of the functions at the intersection equal to the inward or outward limits.

7. Continuous Infinite Valued Logics

It's not necessary to have infinite valued logics satisfy relationships that should only be valid for crisp bivalent logic. The functions defined above satisfy a more general type of the *Law of the Excluded Middle* and the *Law of Noncontradiction*. Differentiation of (51) yields (where prime indicates derivative);

59a) $C'(x) = -1$

59b) $C(x) + x\cdot C'(x)= 0$

59c) $C'[C(x)]\cdot C'(x) = 1$

In (59b) only the derivative of the common simple form, $x\cdot C(x) =xx'= 0$, is used instead of the derivative of (51b). Equations (59a) and (59b) have the solutions

60a) $C(x) = k - x$

60b) $C(x) = k/x$

respectively, where k is a constant of integration. We can relax the constraints given in (59) and still produce logic-like [or infinite-valued or fuzzy logical] results as given below;

61a) $1/2 \le t(x) + t(C(x)) \le 1$ *Extended LEM*

61b) $0 \le x \cdot C(x) < 1/2$

$1/2 \le t(C(x \cdot t(C(x))) \le 1$ *Extended LNC*

61c) $C(C(x)) = x$ *Extended LSI*

The truth value assignments to variables and complementation can also be generalized. We have a function t(x) which assigns a truth value to a variable x, and a generalized complementation/negation operator which operates on x not t(x). So we have $NOT(x) = 1 - x$. Then we'd expect t(x) and NOT(x) to satisfy

62) $t(NOT(x)) = NOT(t(x)) = t(1-x)$

It's easy to see that $t(x)=U(x-1/2)$ satisfies this condition since we have

63a) $t(x) = U(x-1/2)$

63b) $t(NOT(x)) = U(1/2-x) = t(1-x)$

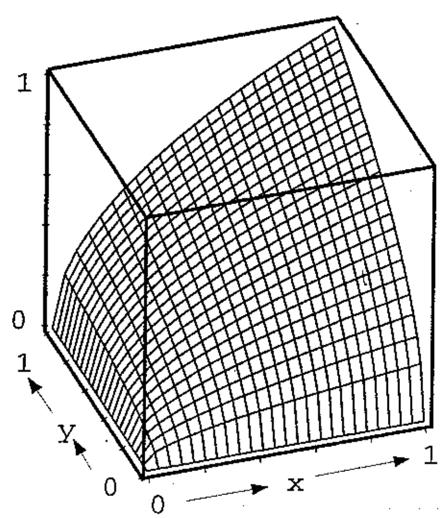
63c) $NOT(t(x)) = NOT(U(x-1/2)) = 1-U(X-1/2) = U(1/2-x)$

8. Simple Product Logic

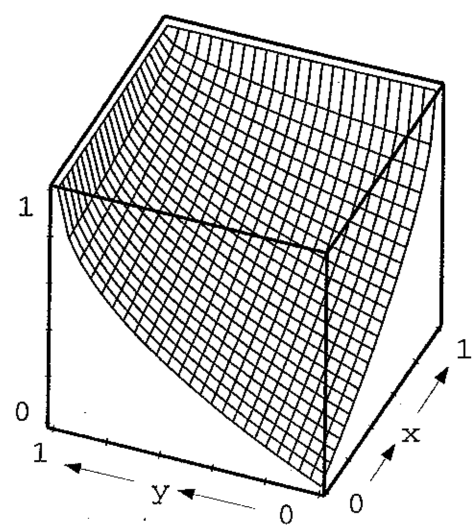
Suppose instead of restricting our attention to unary and binary functions we generalize and then derive the binary functions as special cases of the n-ary functions. All we need to do is to define

64a) $NOT(x)=1-x$

64b) $AND(x_1,x_2,..,x_n) \equiv (x_1 \cdot x_2 \cdots x_n)^{1/n}$



Product AND(x,y)



Product OR(x,y)

The operation of taking the nth root makes the AND operation idempotent. Using the two definitions above we can easily define others using complementation and De Morgan's Laws. For example these definitions are associative

65a) $\mathrm{NAND}(x_1,x_2,..,x_n) \equiv 1-(x_1 \cdot x_2 \cdots x_n)^{1/n}$

65b) $\mathrm{OR}(x_1,x_2,..,x_n) \equiv 1-([1-x_1][1-x_2]\cdots[1-x_n])^{1/n}$

Then for the special case of binary operations these reduce to the simpler forms

66a) $\mathrm{AND}(x,y) \equiv (xy)^{1/2}$

66b) $\mathrm{OR}(x,y) \equiv 1-[(1-x)(1-y)]^{1/2}$

The system is not distributive except at the boundaries since we have

67a) $\mathrm{AND}[x,\mathrm{OR}(y,z)] \equiv [x \cdot [1-[(1-z)(1-y)]^{1/2}]]^{1/2}$

67b) $\mathrm{OR}[\mathrm{AND}(x,y),\mathrm{AND}(x,z)] \equiv 1-[1-(xy)^{1/2}][1-(xz)^{1/2}]$

9. Sigmoidal Truth Valuation

Suppose we use the following sigmoidal [logarithmic] truth valuation S(x)

68a) $S(x) = 1/2 + L(x)/2$

68a) $L(x) = U(x-1/2)G_r(x) + U(1/2-x)G_l(x)$

68b) $G_l = -\ln\{1-\mu[x-1/2]\}/\{\ln(1+\mu/2\}$

68c) $G_r = \ln\{1+\mu[x-1/2]\}/\{\ln(1+\mu/2)\}$

where μ is an arbitrary parameter controlling the steepness of the sigmoidal curve and where we may use the pulse function as developed earlier to make sure that the function is zero outside the interval [0,1]. It can be verified that the sigmoidal function meets the B.C. and $S(1/2)=1/2$ [both the right and left limits are 1/2]. It can be verified that this truth value is involutional; that is $\mathrm{not}(x)=S(1-x)$ and complementing it again, $\mathrm{not}(\mathrm{not}(x)) = S(1-(1-x))=S(x)$. Furthermore since the inflection of the curves L(x) and S(x) get steeper with increasing μ, the step function U(x) is the limit of S(x) as $\mu \rightarrow \infty$.

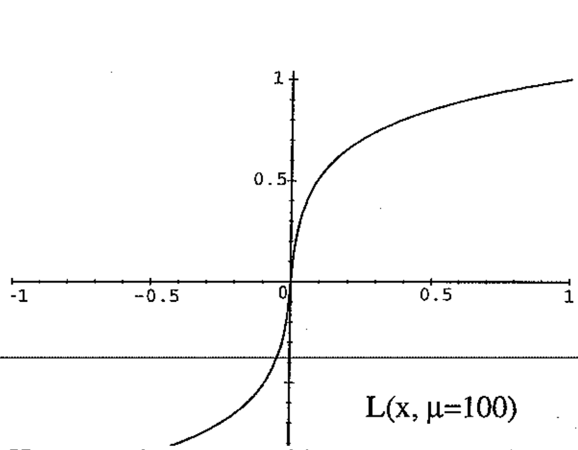
As in the other cases, it is convenient to define AND(x,y) and the complement C(x) functions as

69a) $C(x) = \mathrm{NOT}(x) = S(1-x)$

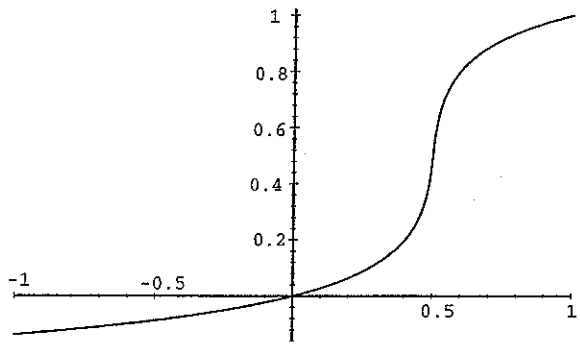
69b) $\mathrm{AND}_1(x,y) = S(x) \cdot S(y)$

69c) $\mathrm{AND}_2(x,y) = S(x \cdot y)$

It can be easily seen that both $\mathrm{AND}_2(x,y)$ and $\mathrm{AND}_1(x,y)$ are associative since $\mathrm{AND}_2(x,y,z)= S(x \cdot y \cdot z)$ or $\mathrm{AND}_1(x,y,z)=S(x) \cdot S(y) \cdot S(z)$.



L(x, μ=100)



S(x, μ=100)

However, they are not idempotent. In order to make them idempotent we'd have to change to definitions as in eqs. (64-66) by taking roots of the products of the sigmoidal truth valuations. From AND(x,y) and NOT(x) we can then define others as in the previous sections;

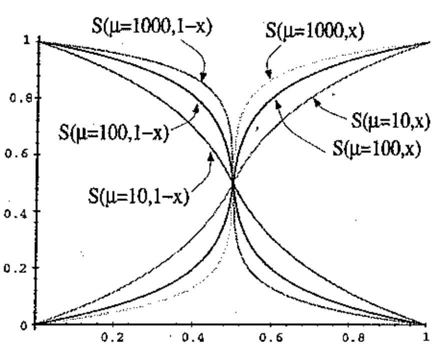
70a) $NAND_1(x,y)= S[1-S(x)\cdot S(y)]$

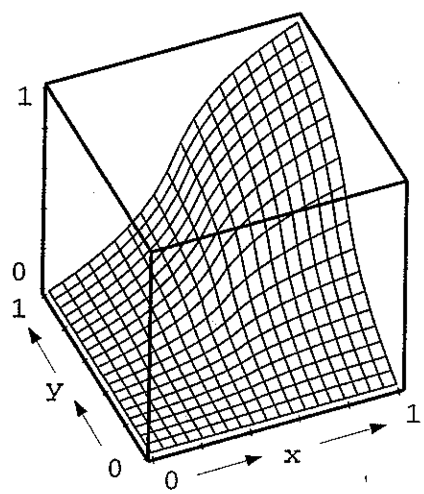
70b) $NAND_2(x,y)= S[1-S(x\cdot y)]$

70c) $OR_1(x,y)= S[1-S[(1-x)(1-y)]]$

70d) $OR_2(x,y)= S[1-S(1-x)\cdot S(1-y)]$

71) $XOR(x,y)= S\{1-S[1-S(1-x)\cdot S(y)]\cdot S[1-S(1-y)\cdot S(x)]\}$





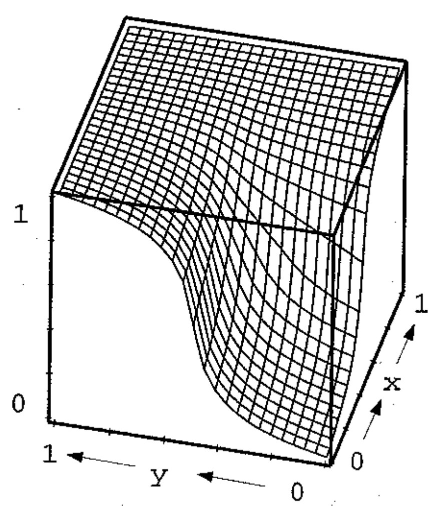
Sigmoidal AND(x,y)

All of the above satisfy the boundary conditions although none of them are idempotent. Distributivity becomes a bigger problem since we now have four possibilities for checking for the validity of $x(y+z) = xy+xz$. We now have different ways in which we can combine the conjunctions and disjunctions as can be seen below;

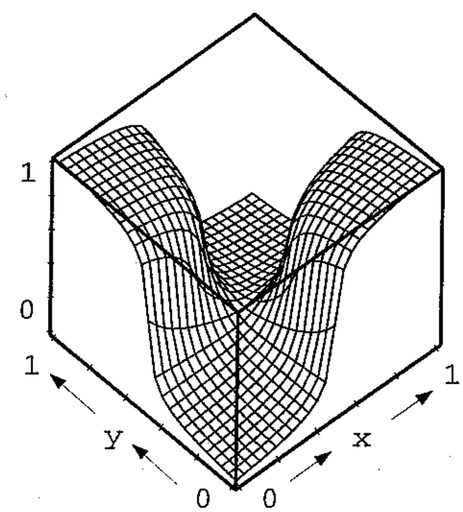
72.1.1.) $x(y+z)= S(x)\cdot S[1-S(1-y)\cdot S(1-z)]$
72.1.2) $x(y+z)= S(x)\cdot S[1-S((1-y)(1-z))]$
72.2.1) $x(y+z)= S\{x\cdot S[1-S(1-y)\cdot S(1-z)]\}$
72.2.2) $x(y+z)= S\{x\cdot S[1-S((1-y)(1-z))]\}$

73.1.1) $xy+xz = S\{1-S[1-S(x)\cdot S(y)]\cdot S[1-S(x)\cdot S(z)]\}$
73.1.2) $xy+xz = S[1-S\{[1-S(x)\cdot S(y)][1-S(x)\cdot S(z)]\}]$
73.2.1) $xy+xz = S\{1-S[1-S(x\cdot y)]\cdot S[1-S(x\cdot z)]\}$
73.2.2) $xy+xz = S\{1-S([1-S(x\cdot y)][1-S(x\cdot z)])\}$

However it can be checked that all of them satisfy distributivity at the boundaries. One of the advantages of the sigmoidal and other truth valuation functions is that meanings can be attached to them and their derivatives as either probability or possibility distributions or densities. For example, if we are performing a forced-binary-discrimination perception test or are attempting to fit a theoretical curve for the results of such a test, we might begin by observing that if we are making the discrimination along a single dimension, say x, such that the valuation $t(x)=1$ implies that it is X, and that $t(x) = 0$ implies that the object is Y [not(X) is a very special interpretation of Y], where for convenience the domain of x has been normalized to [0,1]. For example in linguistics, we may take x to be the VOT [Voice Onset Time] that allows us to distinguish between a voiced plosive /b/ and the unvoiced plosive /p/. Then in forced binary discrimination tests we make a judgement as to whether the phoneme is /b/ or /p/ depending on the value of VOT [other features being the same]. This is equivalent to our perception constituting a mechanism in which we make mental discrimination based on the distance of the stimulus from /p/ [which is 1−x] *and* the distance from /b/ [i.e. x], so that our judgement is based on *both* distances from /b/ and /p/. That is exactly what $t(x)\cdot t(1-x)$ provides us without unnecessarily restricting the form of the function. It would be possible to create a specific kind of perception model by postulating that it varies in particular ways and then derive differential equation models [see Hubey, 1994].



Sigmoidal OR(x,y)



Sigmoidal XOR(x,y)

It would be just as easy to simply pick a form for the function $A(x)= d/dx\ t(x)$, which might be termed the ambiguity function in this case. Its maximum shows the point at which there is a crossover of perception and the point at which the ambiguity, as to whether the object is X or Y, is maximum. No such ambiguity exists at either $x=0$, or $x=1$ where we are certain that the object possesses all the necessary requirements to be not(X) or X. It should be noted that in this case, not(X) is not all things which are not X but something that is very specific so that we can make mental estimates of the distance of the stimulus from this object. In logic, not(X) is anything that is not X, not necessarily something that also exists as a single mental construct. For example, not(apple) is not a single object or a set of objects such as orange or pear. In such a case, the mental computations become more complex and either we are making a mental distinction between an apple and all things which are not apples or between an apple and something which is very close to it. The forced binary discrimination test is a very simplified version of what we do during normal waking hours at every instant.

In a sense A(x) may even be interpreted probabilistically since t(x) would correspond to the cumulative distribution function and hence its derivative to the density function. However, in this case instead of treating the probability density function as the function which has the information necessary to know the probability of an object being X, we instead interpret it as a function disambiguating between two opposing possibilities so that the extreme cases are those in which we have no uncertainty about what the object is, and the maximum problem occurs in the middle [since S(x) is symmetric in the example]. In the case of the earlier truth-functional interpretation of $U(x-1/2)$, we know that its derivative is the Dirac delta function $\delta(x-1/2)$, which is the most extremely peaked distribution, zero everywhere except at 1/2. This corresponds to absolutely no ambiguity anywhere except at the center, and to the right and left of which we are completely certain that one or the other [opposite] situation prevails, which is the standard bivalent logical way of looking at the world. The truth-functional interpretation, then, is flexible and corresponds to degrees of truth or belief. In a sense it is even more primitive [or a priori] than logic since it deals with perception, for example of what is an apple and what is not. Discernment of apples from other fruits is probably done in multidimensional space using a process which may be modeled using fuzzy-logical mathematics as discussed here and elsewhere. Logic takes all objects, apples and non-apples are primitive notions without regard to perception. Hence the fuzzy-logical truth values in some sense are more than logic. Once interpreted this way, then there's no reason why different truth valuations cannot be used on multidimensional problems, such as those that occur in speech recognition since every distinctive feature is not weighted equally by the perception mechanism. Considered in this light, the non-associativity problem may be considered to be useful in some applications since it implies that in certain situations we may be looking for non-commutative binary operations. Furthermore the interpretation of A(x) as an ambiguity function depends on the shape of t(x) as will be shown below.

10. Parabolic Truth Valuation

Define a continuous parabolic reverse-sigmoidal function s(x), in the interval [0,1] via two pieces, $r_1(x)$ in [0,1/2] and $r_2(x)$ in [1/2,1], then we can see that the function P(1/2) will be equal to $r_1(1/2) + r_2(1/2)$ but will not have a discontinuous spike at $x=1/2$ if $r_1(1/2)= r_2(1/2)$. We can derive the parabolic function that meets these criteria. The general parabolic is of form $P(x)=a(x-g)^2+b$. We'd like to construct a piecewise parabolic function to satisfy some of the conditions outlined above. Since the function must meet the B.C. that the function $P(0)= 0$, we see that we must have $a = b/g^2$. Since we'd like this function to also meet the other B.C. i.e. $P(1)= 1$, then $a=(1-b)/(1-g)^2$. From these two we can obtain a relationship between b and g;

74) $$b = g^2/(2g^2-2g+1)$$

Since we'd like the truth value assigment to be symmetric we must choose g=1/2, in which case the values of the other coefficients are a=2, and b=1/2. Therefore the parabolic truth value function is given by

75a) $P(x)=U(1/2-x)(-a(x-d)^2+b) + U(x-1/2)(a(x-d)^2+b)$

which should preferably be written using the pulse function since we want the values to be zero outside the interval [0,1]. Hence

75b) $P(x) = -Q(x;1/4,1/2)(a(x-g)^2+b) + Q(x;3/4,1/2)(a(x-g)^2+b)$

Both (75a) and (75b) give identical results; however one should be mindful of the assumption on the default limits used in the definition of (60b). We can mimic the earlier methods of truth valuations and produce even simpler truth valuations. For example;

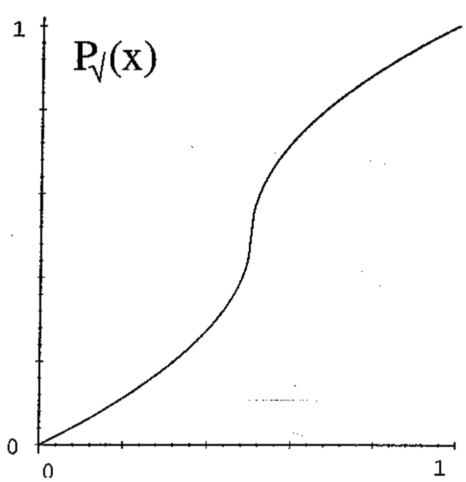
76a) $P_{\surd}(x) = U(x-1/2)\cdot(1+p_r)/2 + U(1/2-x)\cdot(1-p_l)/2$

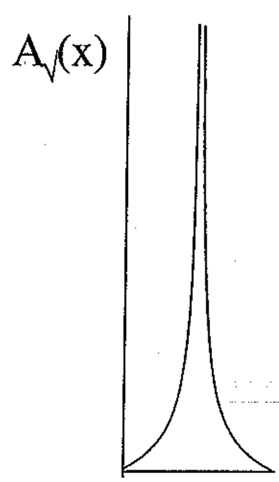
76b) where $p_r = [2x-1]^{1/2}$ and $p_l= [1-2x]^{1/2}$

It can be verified that the right and left limits at 1/2 are 1/2. The limits at x=0, and x=1 are 0 and 1 respectively. The complement, AND, OR and other binary functions can be defined as for the sigmoidal function. It should be noted that NOT(x) is computed as P(1−x) where as before P(x) is a specific kind of truth value assignment to the variable x.

An alternative way of interpreting these results would be to ignore the specific truth assignment function P(x) but to use P(x) in producing the binary connectives of Boolean Algebra in the same manner as U(x) was used and to consider the complement of x as 1−x as is commonly done. It is convenient in this case also to define the AND function as $AND(A,B) \equiv P(AB)$ and then express the other functions in terms of the AND and NOT using De Morgan's Laws. It should be noted that there is no explicit subtraction or addition of 1/2 either in the derivation/definition of the AND(A,B) or the NOT(A) function since this is built-into the function P(A) in the specific assignment of truth values.
Derivative of $P_{\surd}(x)$ i.e. the corresponding ambiguity function is given by

77) $A_{\surd}(x)=\delta(x-1/2)(1+p_r)/2+U(x-1/2)/(2\cdot p_r) +U(1/2-x)/(2\cdot p_l) -\delta(x-1/2)(1+p_l)/2$





Since U(x) was used in (76a) and earlier only for convenience, it's not necessary to include the delta functions in the derivative. However, there still exists effectively a delta function because of the division by zero. It's possible to write $P_{\surd}(x)$

in closed form such that it is zero outside the interval [0,1] without making use of the Heaviside function which is defined in such a way as to be not much more than ad hoc. The general pulse function has been defined in eq. (58). Using it we obtain for the left-half pulse $u_l(x)$ and the right-half pulse $u_r(x)$

78a) $u_l(x) = 1/2 + \{1/2\}\cdot[\{1/16 - (x-1/4)^2\}/\{abs\{1/16 - (x-1/4)^2\}\}]$

78b) $u_r(x) = 1/2 + \{1/2\}\cdot[\{1/16 - (x-3/4)^2\}/\{abs\{1/16 - (x-3/4)^2\}\}]$

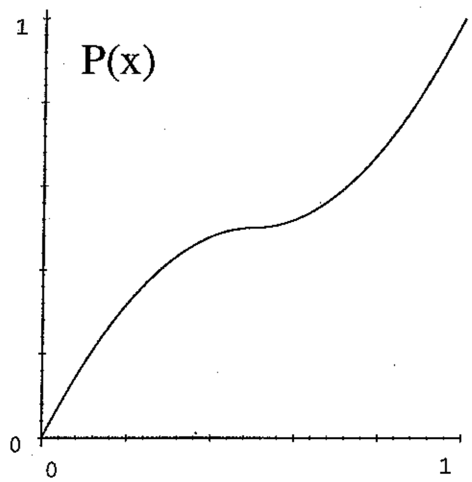
78c) $P_{\surd}(x) = u_l(x)\cdot(1-p_l)/2 + u_r(x)\cdot(1+p_r)/2$

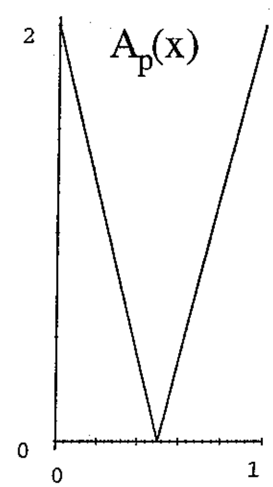
It's not even necessary to always have the truth functions have sigmoidal shapes which are upward concave in the first half of the interval. Since taking the square root of a number $z < 1$ makes the values larger, then $1-z$ becomes smaller and yields the truth valuation as above. However there's no reason why the square root of the parabolic function should be used and not its square. Changing the sets of equations (76) or (78) yields

79a) $P(x) = U(x-1/2)\cdot(1+p_r)/2 + U(1/2-x)\cdot(1-p_l)/2$

79b) where $p_r = [2x-1]^2$ and $p_l = [1-2x]^2$

The plots are shown below and as can be seen A(x) is no longer interpretable as ambiguity.





It is possible to give an interpretation to the derivative of the truth valuation in general. In general, logic as it stands cannot give a faithful description of every natural occurrence. That the quantum phenomena obey a logical calculus which is not distributive is already known. However it's not necessary to appeal to quantum physics to produce quirks in logic. It is also a commonly known fact that if it's called outdoors we can warm our hands by blowing on them. Just the same, we blow on our soup or cup of tea to cool it. So, ignoring the physics for now, if the object is hot, and we blow on it, it becomes cold [not hot] and if the object was cold, then it becomes hot. That this is a true and everyday fact that can be verified experimentally, and has probably been verified by almost everyone at one time. So we now evidently have

80a) $BH \Rightarrow H'$

80b) $BH' \Rightarrow H$

where the prime indicates negation/complementation. It's already looking strange since it resembles the liar paradox, and in fact it's even a better example of something that flip flops in truth values. Therefore we now can show that

81a) $[BH(BH \Rightarrow H')] \Rightarrow H'$

~~81b)~~ $\cancel{[BH'(BH' \Rightarrow H)] \Rightarrow H}$

are tautologies. Everything seems fine, except that the following is also a tautology

82) $[BH(BH \Rightarrow H')] \Rightarrow X$

Since the antecedent of (82) calculates to False, it can imply anything X, which means H or H′ will do for X. In the previous truth valuations, the derivative was interpreted to mean the level of ambiguity at that degree of truth and was suitable for discrimination tests which could in some sense be interpreted as being at the core of bivalent logic. In this case, the derivative is inverted and hence cannot be interpreted as an ambiguity. However, there is a much better interpretation, one that is more suitable for feedback-control theories and applications. In the real world of fluid dynamics and heat transfer via convection, there is thin layer surrounding the hot/cold body [the Prandl boundary layer theory] in which the temperature changes from that of the body to that of the ambient temperature. If there is no convection the boundary layer stays surrounding the body and since air is an excellent insulator, protects the body from rapid temperature changes. If the boundary layer is swept away by convection [as on windy days, when the so-called wind-chill factor comes into play], then there is much more rapid rate of heat transfer. Since heat exchange takes place from the hotter to the colder body, it's perfectly understandable why in the case of a hot cup or teaspoonful of tea, the air convection cools it (bringing it down to the ambient temperature) and on a cold winter day, warm breath increases the temperature of the hands. In such a case, the effect [that is, the rate of heat transfer] is greatest at the two extremes, when the differences between the temperature of the object and the air blowing across it are the greatest and zero when they are at the same temperature.

This kind of an effect generally holds in control applications; the greatest effect is felt or should be felt furthest from the target [the control parameter] and there's practically no effect if the target state and the actual state of the system are very close or equal. It would seem that control applications and discrimination applications obey different types of truth valuations and probably should obey different types of effect functions. In one type of "fuzzy logic" the midpoint is stable, in the other the stability is at the extremes. Bivalent logic is the case in which we seek stability at the extremes [i.e. true or false] and we strive not to produce any value anywhere in the middle [as in the liar paradox]. In the control applications, stability is toward the midpoint, and it is the extremes that are unstable.

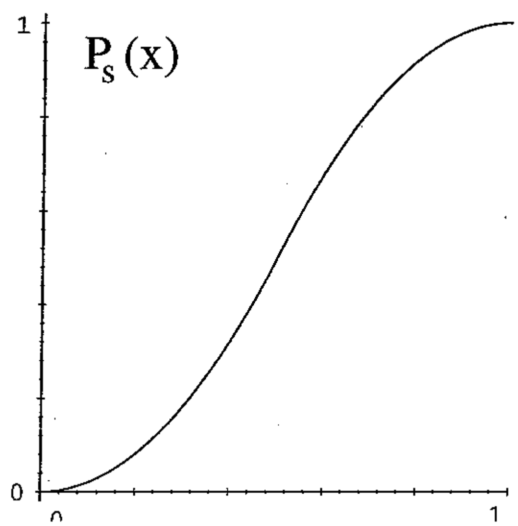
In general there is a control parameter u= (Actual Value–Target Value), where u can be positive or negative, say to the normalized interval $[-1,+1]$. In the cases above, this interval has been mapped to $[0,1/2]$ so that there is a simple relationship between the two different types of truth-valuations above; one of the effect/ambiguity functions is circularly shifted by half the interval, a kind of a modulo arithmetic. In the case of the truth functions, there is a double symmetry; they are shifted circularly as in the case of the effect/ambiguity functions but also circularly shifted upwards and downwards. In fact, in infinite-valued logic it's easy to see that the XOR function is essentially almost a saddle-point instability.

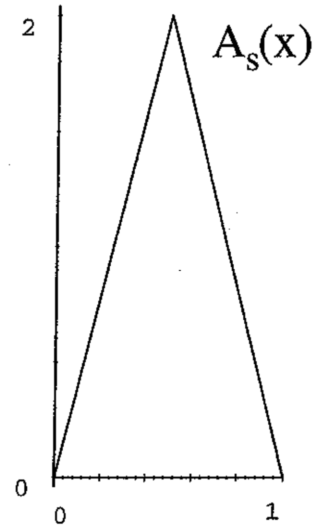
A very *smooth* parabolic truth valuation can be produced by

83a) $P_s(x)=2x^2\cdot U(1/2-x) - (2[x-1]^2+1)\cdot U(x-1/2)$

83b) $\{d/dx\}P_s(x)= 4x\cdot U(1/2-x)+(4-4x)\cdot U(x-1/2)$

where the delta function has been ignored in the calculation of the derivative. The $P_s(x)$is a piecewise constructed function but it can be verified that the both the function and its derivative are continuous at 1/2. The valuation and its derivative are shown on the plots below.





It should be noted that what we'd normally considered as NOT(x) can quite easily serve as a truth value in situations involving human psychology. Some of us humans, being fond of thinking and priding ourselves in our real or imagined intelligence, and not having the opportunity to indulge ourselves in scientific research might find everyday stories more believable, the less of the story is told. Especially those with the border-line paranoia are more easily fooled if only enough of a story is told for them from which to draw conclusions. In order to rile such a man, it's only necessary to show a photo of his wife or ex-wife with one of his friends and let his intelligence and imagination [or paranoia] do the rest. In this case, the greatest response is obtained for the smallest inputs whereas outright accusations might be met with incredulity. In some cases [maybe most] some people might be prone to apply the perverted version of the golden rule; "Do onto them before they do onto you!' and thus gaining even more enemies and making their outlook on life a self-fulfilling prophecy.

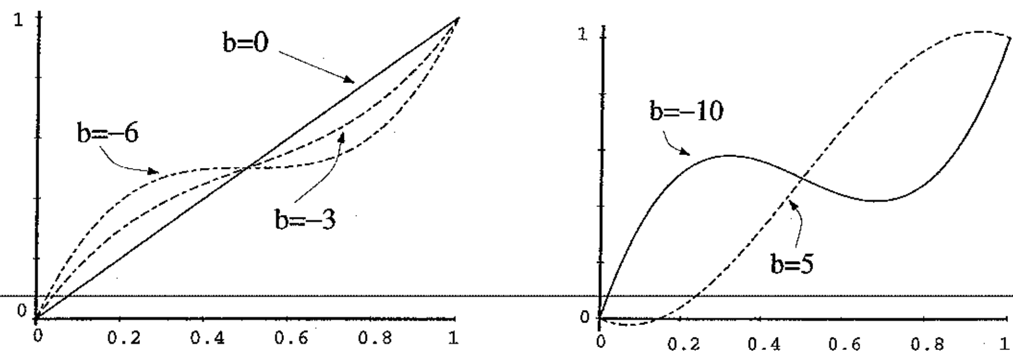
11. Cubic Valuation

A non-piecewise cubic function that meets the requirements of the previous sections is given by

84a) $K(x)= -2bx^3/3 + bx^2 + (1-b/3)x$

84b) $K(1-x) = 2bx^3/3 - bx^2 + (b/3-1)x + 1$

where $-6 \le b \le 0$. It can easily be verified that valuation is involutive. For b=0 K(x) reduces to the linear valuation. For values $0 < b < -6$, K(x) is not monotonic.

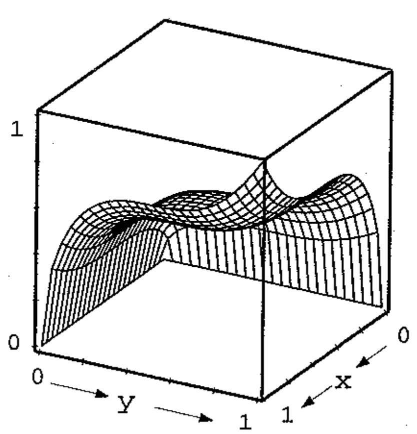


Cubic Valuation K(x)

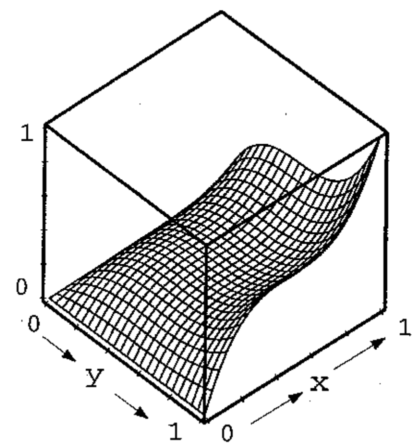
As in the earlier examples, there are various ways of defining the usual binary connectives. In these cases it is much easier to first define the AND function and the standard negation/complementation. Thus we have

85a) $\mathrm{AND}_1(x,y) = K(x)\cdot K(y)$

85b) $\mathrm{AND}_2(x,y) = [K(x)\cdot K(y)]^{1/2}$

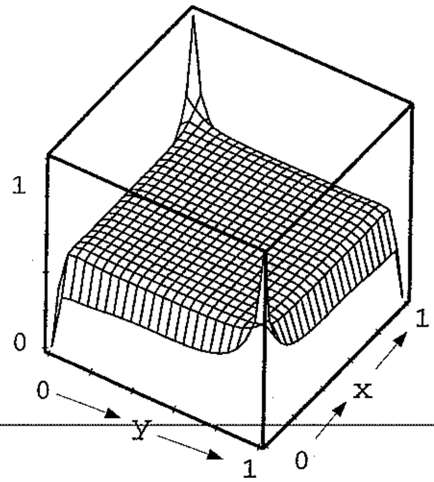


Cubic $\mathrm{AND}_1(x,y)=[K(x)K(y)]$

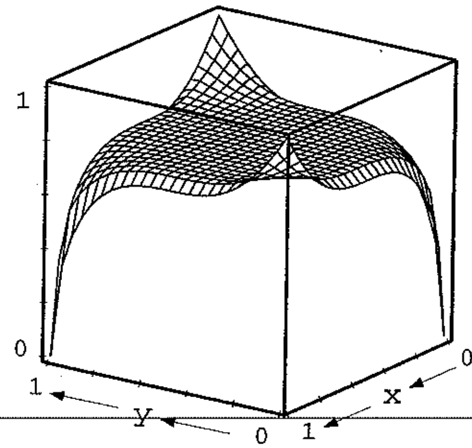


Cubic $\mathrm{AND}_2(x,y)=[K(x)K(y)]^{1/2}$

The definition of $\mathrm{AND}_2(x,y)$ as above makes it idempotent as in the simple produc logic as defined earlier. It's clear that (85b) should be treated as a specialization of a more general n-ary AND function, as in section 11, so that we don't have problems with associativity and idempotency. In other applications nonidempotent AND functions might be useful since there might be cases in which we might want the multiplicative interactions of two variables to be smaller or larger than for the case of a single variable. Straddling the transition zone between probability theory and logic, as infinite valued logic does there is leeway for interpretation of the connectives or the functions of this logic. For example, in the famous prey-predator interaction equations, also known as the Lotka-Volterra equations we have



Cubic $XOR_2(x,y)$



Cubic $XOR_1(x,y)$

86a) $dp/dt \propto p - pr$

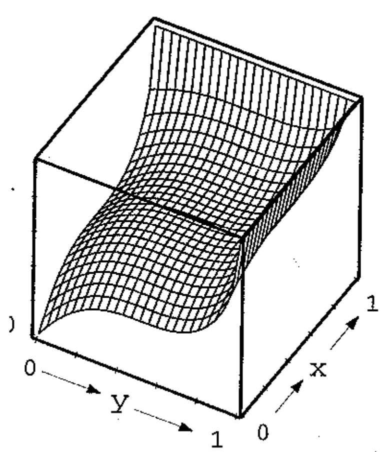
86b) $dr/dt \propto -r + pr$

The prey (p) increase if there are no predators (r) and the rate of increase is negatively affected by the interaction of prey *and* predators (the pr term) in which it is clear that the pr term corresponds to an AND function. Similarly the predators decrease if left alone since there is no food source for them, and they increase if both prey *and* predators exist. These equations, in effect, are nonlinear feedback equations and since they produce nonlinear-oscillations, they may be thought of as analogs of the linear harmonic oscillator or the liar paradox.

From the definition of AND we can define an associative and idempotent OR(x,y)

87) $OR(x,y) = 1 - AND[NOT(x) \cdot NOT(y)] = 1 - AND[K(1-x) \cdot K(1-y)]$

There are other possibilities but similar to the constructs of the previous sections but are not shown. The exclusive OR's have interesting properties as shown below.



(x,y)=1−AND [K(1−x)K(1−y)]

88a) $XOR_1(x,y)=1-(1-[K(x)K(1-y)]^{1/2})\cdot(1-[K(y)K(1-x)]^{1/2})$
88b) $XOR_2(x,y)=1-\{K(1-[K(x)K(1-y)]^{1/2})\cdot K(1-[(K(y)K(1\text{-}x)]^{1/2})\}^{1/2}$

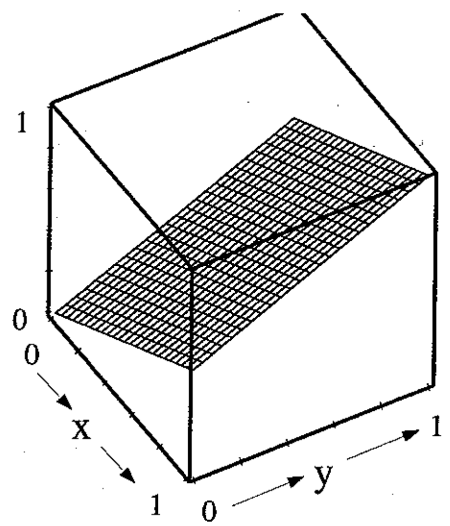
These equations are clearly more like the standard saddle-points except that the stable regions and their stability characteristics are clearly much different.

12. Generalized Idempotent and Continuous Max-Min Operators

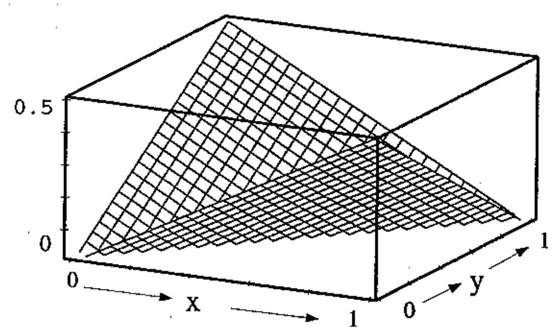
Consider the functions given below

89) $H_h(x,y) = (x+ y)^h/ 2$
90) $M_m(x,y) = 2^{(m-1)} \{[(x-y)^2]/[2((x-y)^2)^{1/2}]\}^m$

It's easy to see that the $H_1(x,y)$ takes one as its maximum value, and zero for its minimum for $0 \leq x,y \leq 1$. In addition, for h=1 it's linear in both x and y. The function $M_1(x,y)$ is always positive, symmetric and zero for x= y. It's essentially $(x-y)^2/\text{abs}[2(x-y)^2]$ or abs(x−y)/2 since the positive square root is taken in the denominator by convention. Hence the elementary operations are being used to define the abs(·) function instead of defining abs(·) as a primitive operation or a primitive function. The plots for these functions for some values of the parameters m and h are shown below.

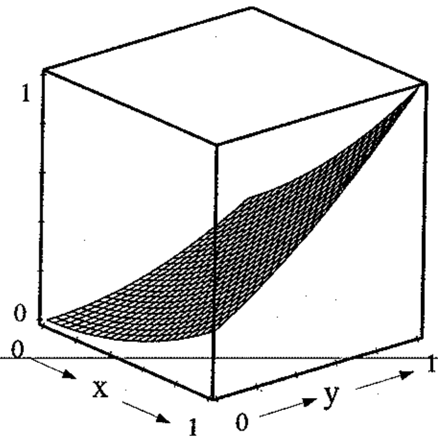


$H_1(x,y)$

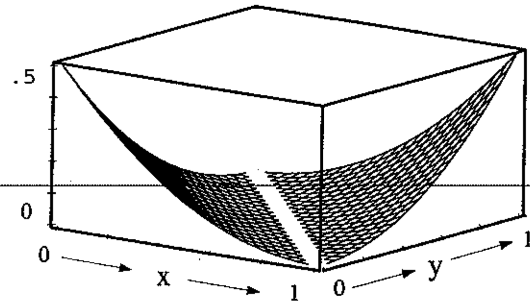


$M_1(x,y)$

It can be seen that H1 is a plane and that M1 is not smooth because of the discontinuity of the derivates along x=y. However, the algebraic sum and differences of these functions are exacctly Max(x,y) and Min(x,y). It's easy to verify that



$H_2(x,y)$



$M_2(x,y)$

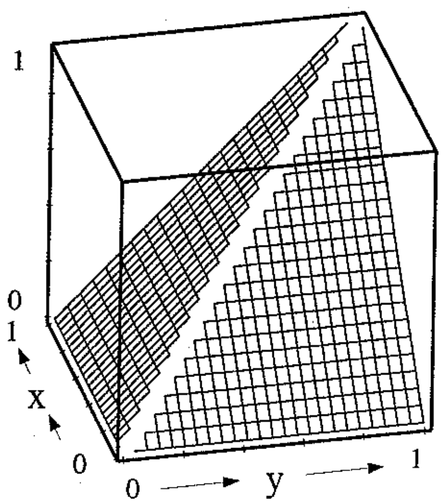
91a) $\quad \text{Max}(x,y) = V_1(x,y) = H_1(x,y) + M_1(x,y)$

91b) $\quad \text{Min}(x,y) = A_1(x,y) = H_1(x,y) - M_1(x,y)$

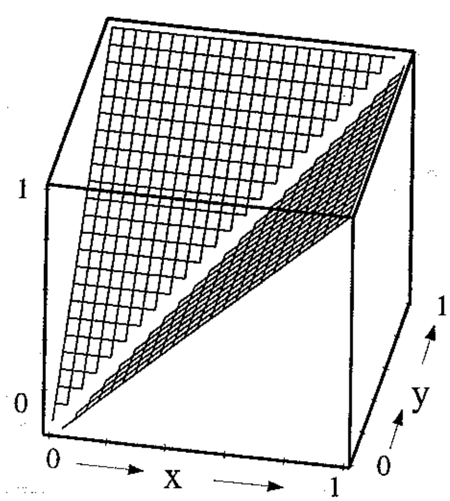
It can be seen that Max(x,y) and Min(x,y) are not the only continuous and idem potent functions that implement union or intersection. In particular, since $M_m(x,y)$ is zero along the diagonal (i.e. x=y) and $H_1(x,y)$ increases linearly along the diagonal from zero to one (i.e. is idempotent) then all functions of form

92a) $\quad V_m(x,y) = OR_m(x,y) = H_1(x,y) + M_m(x,y)$

92b) $\quad A_m(x,y) = AND_m(x,y) = H_1(x,y) - M_m(x,y)$



$H_1(x,y) - M_1(x,y)$



$H_1(x,y) + M_1(x,y)$

are idempotent and continuous since the form of $H_1(x,y)$ guarantees it , but only the max-min functions as defined in eqs. (91) are also associative. Analogous results can be obtained for other values of the parameters m, and h. It can be seen that $V_2(x,y)$ and $A_2(x,y)$ do not satisfy the crisp B.C. along the edges but only at the points corresponding to the binary values {0,1}.

Other binary connectives and even nonstandard logic-like functions can be derived from the basic functions $H_h(x,y)$ and $M_m(x,y)$. For example the functions

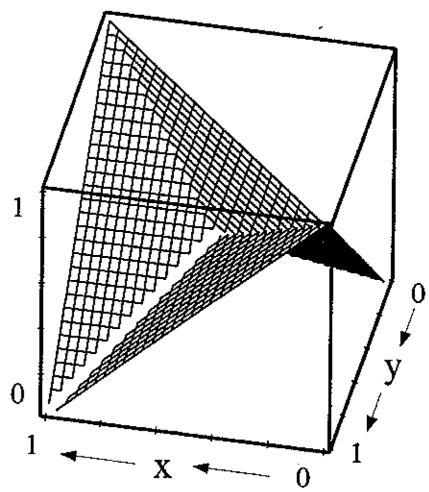
93a) $U_n(x,y) = [H_1(x,y) + M_1(x,y)]^n$

93b) $I_n(x,y) = [H_1(x,y) - M_1(x,y)]^n$

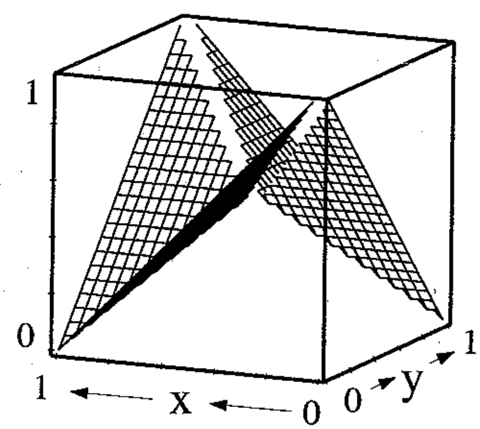
are continuous but not idempotent and satisfy the boundary conditions not only at the end points but along the *edges*. By dropping the idempotency requirement even more flexibility is gained. For instance, it is well known that for independent events A and B, we have

94a) $P(A \cup B) = P(A) \cup P(B) = P(A) + P(B)$

94b) $P(A \cap B) = P(A) \cap P(B) = P(A)P(B)$



XOR(x,y)



EQ(x,y)

However no one ever attempts to substitute A for B and then derive

95a) $P(A \cup A)=P(A)=P(A)+P(A)=2P(A)$

95b) $P(A \cap A)=P(A)=P(A)P(A)=[P(A)]^2$

We simply '*know*' that it should not be done; in other words, instead of requiring that the measures over these sets be explicitly defined in such a way as to satisfy idempotency we resort to the '*handwaving*' solution. It seems that fuzzy logic or more correctly fuzzy sets and measures are filling an important gap;

the transition zone between probability theory and logic. Continuing with the V(x,y) and A(x,y) we can express XOR(x,y) and EQ(x,y) as

96a) $XOR(x,y) = 1 - A\{[1-A(1-x,y)],[1-A(x,1-y)]\}$

96b) $XOR(x,y) = V\{A(1-x,y),A(x,1-y)\}$

96c) $EQ(x,y) = V\{A(x,y),A(1-x,1-y)\}$

Other combinations of the H and M which are idempotent can be produced easily but instead of satisfying the boundary conditions along the edges, they might satisfy them only at the end points i.e. the corners. What is of greater interest for A(x,y) and V(x,y) is that being continuous, they are differentiable and on the basis of this we can derive other unknown relationships among these functions. Since the definitions are in terms of polynomials both the differentiation and the integration of these functions are straightforward. More fuzzy logics can be found in Hubey[1997].

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