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Dear Professor Fetzer

Please find enclosed four copies each of the two manuscripts titled "The Time-Space Super-Liar, Consistency-Decidability, Oscillation, and Other Properties" and "Re-run of the Liar, Falsificationism, and ω -inconsistency". Please accept my apologies for not providing copies last time.

Respectfully,

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Re-run of the Liar, Falsificationism, and ω -inconsistency
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Re-run of the Liar, Falsificationism, and ω -inconsistency
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Abstract

Starting from the Hempel Paradox, the implication is re-interpreted so that it is shown that the standard True/False dichotomy should be interpreted as *Provisionally True-Definitely False*, which unlike the quantifiers $\{\forall, \exists\}$ and $\{\text{Necessarily, Possibly}\}$ is already implied by the assignments of truth values to the implication operator. From this point of view, the inductive problem is everywhere (since axioms are, in truth nothing more than thinly disguised inductive generalizations), and that deductive analysis is true only relatively speaking and on an absolute scale a myth. Furthermore the standard argument over whether the theory determines what we can measure or whether we gather data/information and from which we produce theories is also resolved on the basis of the fact that deduction is done only on a portion of the whole which we can never comprehend except inductively, and that the scientific process consists of alternating between deductive and inductive modes at different scales.

It is shown that the standard set theoretic infinities are the real reasons for using the symbols \forall , and \exists instead of \prod , and \sum . Once the latter set of symbols is used ω -inconsistency disappears. The Liar Paradox, and the way statements are normally composed using the symbolism of propositional logic (or suppressed quantifier notation) is shown to lead from possibly an incorrect or unclear usage and that the standard composition can be changed to produce yet another 'solution' of the Liar Paradox, and also the Truth-teller Paradox.

In the same vein, logic is reinterpreted to show that oscillation is a property which logic has and that trying to force it to stop is producing the interminable disagreements over the meaning of diagonality. The simultaneous solution of a pair of equations, in which the values are explicitly assigned to variables like in other branches of mathematics clarifies the problem of the oscillation of which various logical paradoxes are specific instantiations. Other paradoxes are also discussed from the same point of view in which the oscillation (chaos in logic) is natural and expected. Explicit assignment and simultaneous solution attempts similar to linear algebraic methods is also shown, and analogues of the same operation in other related fields is shown. Explicit assignment of values to the variables are used to clarify the problems of the fixed-point oscillation, and simultaneous solutions via linear-algebra like and differential equation methods are shown to indicate possible solutions to the problem of the fixed-point oscillation.

1. Introduction: The Raven Paradox and Implication

One particular problem in logic which has been a perennial thorn in the side of philosophers of science and 'philosophical' logic has been the role of implication and the diagonal proofs (or paradoxes) of infinite set theory. A particular problem has been the role of logic in the philosophy of science, especially such ideas as positivism, confirmationism, and falsifiability. Since these fields can also be treated with probability theory, and since the development of non-bivalent logics to deal with such paradoxes and lately of infinite-valued logics (the so-called fuzzy logics) questions about the relationship of logic to probability theory or to fuzzy logic or the relationship of fuzzy logic to probability theory and also to related topics such as the nature of intelligence have given rise to many popular and formal treatments of the subjects. The purpose of these chapters is to display some original relationships, including interpretations of the implication of logic, and production of many multiple valued logics, as well as display the much more explicit nature of the universal and existential quantifiers, and show why standard propositional logic has many things already built into it.

The discussion starts with the classic Raven Paradox of Hempel, and the particular truth values assigned to the implication, the interpretation of the assignment of such values, and a reinterpretation of the diagonal paradoxes, and simultaneous conditions implied in logic.

$P=1$ and $Q=0$: The only one we can be sure of is this assignment of truth values since then the assertion $P \Rightarrow Q$ (whenever P is true, then Q is true) must be false. It's obvious that this must be the case. For example, let

- 1) $P \equiv$ It's a raven and $Q \equiv$ It's black.

Then $P \Rightarrow Q$ means "If it's a raven, then it's black." If it's a raven ($P=1$) and it's not black ($Q=0$), then the statement "If it's a raven, then it's black" can't be true; hence ($P \Rightarrow Q$) must be assigned 0 for this particular case. So far, no problems.

$P=1$ and $Q=1$: What would we assign for this? In other words:

- 2) If it's a raven ($P=1$) and it's black ($Q=1$), then what is the value of ($P \Rightarrow Q$)?

Can we assign the value 1 (True)? If we've seen a single raven and it's black, can we then assign the value T (true) to the proposition ($P \Rightarrow Q$)? Well, strangely enough, that's exactly the value that gets assigned. Now we have the strange paradox that the tool of deductive logic seems to be based on induction (of the worst kind). Do we have some justification for this truth assignment? It's not so clear that it's correct if we consider that we are using this assignment as a guide in an experimental search for truth. It's not so clear but a justification can be made for it. After all you can say that if you see enough ravens ($P=1$) and all are black ($Q=1$) then the implication, $P \Rightarrow Q$ should be assigned a value of True. Sure, but how many ravens do we have to see before we know this is true. Come what may we still can't escape the induction, and the darling of the deductionists is based on induction. There's no other kind of reasoning. Everything in the final analysis is based on inductive reasoning, including, of course, the laws of deductive logic.

The concept of *a prioriness* is a rather disingenuous escape from this simple truth.

On the other hand, if we work backwards in an attempt to justify this particular assignment, then it says that the original implication is somehow already known (how?) to be true since the question is not even about seeing ravens at this point. The formula is general. It says that $(P \Rightarrow Q) = 1$ for $P=1$ and $Q=1$. It doesn't say that you have seen any more than one raven. It doesn't say you have seen two ravens. It doesn't say anything about how many ravens. It sounds like it works because every time you substitute something sensible, it gives a sensible answer. But like *I Ching*, it gives nonsense answers when nonsense is substituted. In more modern sense it is GIGO [Garbage In-Garbage Out] and it really can't be any other way.

3) Suppose we let $P \equiv$ It's rose and $Q \equiv$ It's red.

Then it says, that if it's a rose ($P=1$) and it's red ($Q=1$) then $P \Rightarrow Q$ is true (i.e. If it's a rose, then it's red). It sounds just like the raven problem. But we know very well that this isn't true at all. Some roses are black and some are yellow. So this means that this assignment is true if the implication is true but it's not true if the implication is not true. There's no way of escaping this conclusion.

$P=0$ and $Q=1$: It's very clear that if we assign True to this case, the conditional is not causal, or at least P is not the sole cause of Q even if we attempt to provide some kind of a causal interpretation.

$P=0$ and $Q=0$: It gets even worse before it gets better. Let's try;

4) Let $P \equiv$ I fall in the lake. $Q \equiv$ I get wet.

Now we have; 'I haven't fallen in the lake ($P=0$) and I am not wet ($Q=0$), then what value should the statement "If I fall in the lake, then I will get wet" be assigned?' i.e. what value should be assigned to $[P \Rightarrow Q] = ?$ Why, we should assign the value true (1) since we know we'll get wet if we jumped into the lake! After all, if you didn't jump into the lake, surely you won't get wet, and we know that you would have gotten wet if you did jump in the lake. And then I would say, "Surely you jest!" Of course, we "know" you'll get wet if you jump into the lake. What's that got to do with confirming the implication $P \Rightarrow Q$? That means it'll work only when it works! In other words, it's true only when it's true? Note that the interpretation "*P is sufficient for Q*" is a very good one, for we know that "falling in the lake" is sufficient to cause "wetness" but only in this specific case since we already know it (how?!). The other part, "*Q is necessary for P*" seems to say "Being wet is necessary for proving the fact of falling in the lake." It seems to work better if we use the contrapositive and say that "Not being wet is sufficient to show that I could not have fallen in the lake." If we go back to the ravens; I have something in my hand and; it's not a raven ($P=0$) and it's not black ($Q=0$) and then can I still assign the value true to the proposition "If raven, then black" i.e. $(P \Rightarrow Q) = 1$ This knowledge seems to have come from some other place. Surely, it's true if it's true and it's not true if it's not. So then, what exactly is implication implying? What value should one assign in such a case of no knowledge about anything? Surprise, surprise, surprise! It's assigned a value of true! But, one might object, of course, that's what should be assigned; after all we just demonstrated its truth! Let's try it again. But this time let;

5) $P \equiv$ It's pleen. $Q \equiv$ It's quue.

Now what value would we assign to the proposition "If it's pleen, then it's quue" when we have seen *nothing* that's a pleen ($P=0$) and *nothing* that's a quue ($Q=0$)? We can't decide because we have no idea what either pleen or quue is? This is Hempel's Raven Paradox. Actually they're related but the result of

the Raven Paradox is that the implication itself \Rightarrow is being questioned (for some people) and the definition of implication is being clarified (for others).

The discussion of the assignment of truth values for various cases has been motivated not only by an attempt to understand what the conditional means but also by Hempel's Raven Paradox which was originally a counterexample to demonstrate the problems of the confirmation theory of science and essentially says that since $N \Rightarrow A$ (If raven, then black) is equivalent to $(\bar{A} \Rightarrow \bar{N})$, if we claim that every time we see a raven that's black we are confirming the proposition $N \Rightarrow A$, then every time we see a thing that's neither black nor a raven, then we are similarly confirming the proposition that ravens are black. So what happens here? Allegedly the most fundamental tool of deduction (formal logic) itself is either apparently based on inductive reasoning or is some kind of a joke when it comes to its axioms? How? Well, that means that if I see a white swan, a yellow banana, or even a red Maserati, I can say this confirms "ravens are black" since none of these cases has anything to do with raven ($N=0$) or being black ($A=0$). In other words; "Ravens are black" [is true] only if it's true that ravens are really black. It seems as if logic is ideally suited to "proving" the most ridiculous things by playing into semantic tricks. It works only when it works. In other words, the implicational proposition ($P \Rightarrow Q$) must already be true, since we assign the value of true to every combination except $P=1$ and $Q=0$ (except when it is false). But if it's already assumed to be true why then do we have to say it's false when it's false? In other words, if we can use it only if the assertion is already true; which really means, it's true only if it's true, then how can $P=1$ and $Q=0$ happen? Surely, it's just a completion of the statement "it's true only if it's true" since it's the reverse side of the coin. The addition of this particular assignment to the meaning of $P \Rightarrow Q$ says "It's true unless it's false." That is we assign the value true to all cases, unless we already know it to be false ($P=1, Q=0$). So propositional logic is biased in the sense that the assertions of the type $P \Rightarrow Q$ are [assumed] true unless they are [can be proven to be] false. Can we say that it also says "the assertions are false unless they are true"?

The real stretching of the imagination (and also the real work of convincing anyone) is accomplished via some other unspecified means, sometimes disingenuously called *a priori*, sometimes *mathematical or set theoretic intuition*. If we don't have any idea what we're talking about, we can't tell if what we proved is true i.e. Suppose $P=pleen$ and $Q=qrue$? Someone might know what these mean so he might think it's reasonable to propose $P \& (P \Rightarrow Q) \Rightarrow Q$. How would you be able to object? That's the major problem and that's the reason formal logic is pretty much useless in discovering the secrets of the universe even if the universe consisted only of mathematical objects. Truth (whatever it is) cannot be discovered with formal logic since it works only when it works and it doesn't work when it doesn't. But that's essentially what Tarski said.

The introduction of the existential quantifier and changing the symbolism allows us to make more sophisticated (complex) statements but its fundamental weakness is always there and thus is the cause of not only the disaffection with it but also the cause of the search for multiple valued logics and even explains the surprising success of fuzzy logics. The predicate symbols are there essentially only to facilitate the book-keeping, much like the dx in integration. Bivalent logic will always be as weak as its weakest link, which is the bias in the conditional toward truth; innocent until proven guilty or true until proven to be false.

Despite the problems with the definition of material implication, there doesn't seem to be much more that can be done at the binary level. The only thing left to turn this particular sow's ear into a silk purse seems to be to put additional restrictions on how the connectives may be combined together to form expressions which maybe assigned truth values. According to the particular rules (inferencing schemes) and the initial truths (axioms) some simple things are extremely difficult to prove, and sometimes what can be proven in one system cannot be proven in another. It seems to pretty much have reached its limit. It seems to be the

camel that was supposed to have been the proverbial horse that was designed by the proverbial committee. In what follows algebraic manipulations are almost always used to bring this back to mathematical roots instead of the medieval philosophy which seems to have infected it. This last desperate move of mathematical logic away from set theory to the theory of formal languages will be no more successful than exclaiming set theory to be the foundations of mathematics; the Holy Grail will always be beyond our reach. Set theory combined with measure theory is not much more than a cheap abstraction and shameless reverse engineering of the most successful branch of mathematics; that is what is now known as analysis. And it's unlikely that set theory's truths will withstand the test of time as to being any kind of a foundations. It will more and more into feverish discussions involving *psychology, cognition, thinking, intelligence and understanding*.

In a discussion of the meaning of implication in logic Rescher [1969, p. 166] states that beginners often experience difficulties and in an effort to explain the assignments of truth values to implication, and that; "It is relatively unproblematic to lead the explanation to the point." where the table given is shown in the figure below. Furthermore he discusses various other possibilities (alternative implications) for the blank spots using quasi-truth-functional systems (for example onedue to Reichenbach). The only assignment of which we can be sure of is line III in the Figure I. It says that if $P=1$ and $Q=0$, then the assertion $P \Rightarrow Q$ must be false. No matter from which school of logic, no one ever argues that if a counter example can be found for a general statement then the statement is not disproved. To justify cases I and IV we should probably argue that if we accept that $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$, then the definition of "whenever P, then Q" makes $P \Rightarrow Q$ true for $P=1$ and $Q=1$ which by symmetry via the contrapositive implies that the case for $P=0$ and $Q=0$ should/could also be assigned the truth value of true. Although this case is also arguable via the Raven Paradox of Hempel, we can still trample over the objections by pointing out the cases in which the contrapositive makes some *sense* (whatever it means).

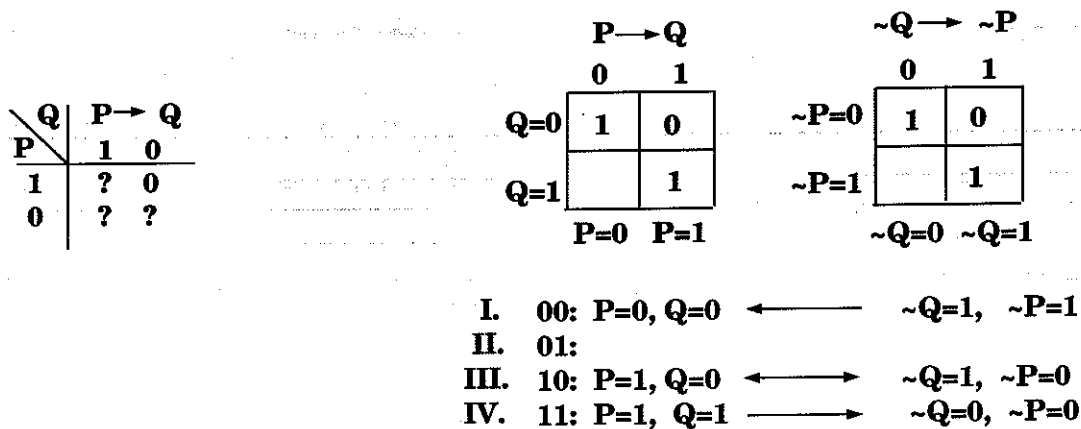


Figure I

But we're still stuck with Case II; why should it be assigned the truth value of true? Gamut [p.33, 1991] makes a similar comment that assigning a true value at this point "doesn't seem very attractive to say that it must always be true." However his reasoning is that; "Since we have agreed that indicative sentences are either true or false, let us choose the least unattractive alternative and say that material conditionals are true if their antecedent is untrue." We normally get around this apparent paradox (such as the Hempel Raven Paradox) by defining the implication to mean that there's no causality between P and Q, hence the implication is simply one-sided; that is, Q can occur without P but must occur if P occurs. A much simpler and intuitive interpretation is given below for the implication, truth and falsity.

If instead of trying to justify all the truth assignments in the implication table, suppose we simply accepted that the only kind of an *absolute proof* is proof of the falsity of a generalization and that the counter example to the general statement is the only acceptable method of proof of falsity. In one stroke we can do away with the objections of the constructivists since the counter-example is a constructive or demonstrative proof. Actually, it would be more appropriate to call it a *destructive proof*. Now that we have a definite definition of falsehood, we can then define truth as not false; that is, if something cannot be proven (via a counter example) to be *definitely false* then it must be true. However, since we have defined everything not [yet] proven via a counter example to be false as true, then the implication is biased toward truth; that is, the truth value is provisional. Hence we have *definite falsity* and *provisional truth*. Thus something is provisionally true provided that we have not yet been able to prove it (via a counter example) to be definitely false. This interpretation is closer to how we really behave and how science is really done. Science has a bias; like time and entropy it has a preferred direction. Science cannot exist without generalizations; every generalization we make is provisionally true and subject to falsification in the future. Every time a falsification occurs, we shrink the domain of applicability of a scientific theory. This way of looking at the implication also leaves wide open the possibility of multivalent logics since we now don't always have to define everything that is not (definitely) false to be true but can make up other values such as *unknown, unknowable, indefinite, or meaningless*. We can even have degrees of truth as in probabilistic logic or fuzzy logic. This would also go some way toward easing the *problem of induction* since this definition explicitly states that there is no absolute truth only absolute falsity. Since in theory it would take an infinite number of observations to conclusively decide that all ravens are black or an infinite number of experiments to "prove" that Newton's Laws are valid or an infinite number of additions of integers to prove that $x+y=y+x$ is true, we can leave the truth value to be provisionally true or accept bivalent *standard classical logic* as an approximation of an infinite-valued logic in which the *infinitesimal value* (in a figurative manner) only may be considered to be absolute truth. It's a cheap way of getting rid of the problem of definition, about as good as Dedekind-Peano fifth axiom.

2. Syllogisms, Quantification, Algebraic Logic, and ω -inconsistency

Leibniz and Venn's ideas using sets to develop the theory of syllogisms based on an algebra of terms produced four kinds of statements [see for example, Tiles, 1989]. The traditional typology using a letter to designate the various forms is affixed to the type of statement. This typology of basic forms may be classified according to: Quality vs Quantity where quality is *Affirmative* or *Negative*; and quantity is the Particular and the Universal as [see for example Nolt & Rohatyn, 1988]

TABLE I

<u>Statement</u>	<u>Type</u>	<u>Coding</u>	<u>Quality</u>	<u>Quantity</u>
<i>All S are P</i>	A	$S \cdot (\text{not}_P) = 0$	<i>Affirmative</i>	<i>Universal</i>
<i>Some S are P.</i>	I	$S \cdot P \neq 0$	<i>Affirmative</i>	<i>Particular</i>
<i>No S are P.</i>	E	$S \cdot P = 0$	<i>Negative</i>	<i>Universal</i>
<i>Some S are not P.</i>	O	$S \cdot (\text{not}_P) \neq 0$	<i>Negative</i>	<i>Particular</i>

It's easy to see that these categorical statements can be written quite simply using the existing machinery of classical mathematics.

$$\textcircled{1} \text{ All } S \text{ are } P. \quad \prod(S_i \Rightarrow P_i) = \prod(S_i' + P_i) = \prod(S_i P_i)' = 1$$

It's obvious why this must be so using the definitions of; (a) implication, (b) of multiplication as *AND*, and (c) 1 as *True*. The statement says that each of the *i*th [S is P] is true or that each factor is equal to 1 so that the product must equal 1. If we complement both sides of $\textcircled{1}$, the equality still holds, so availing ourselves of De Morgan's Law we obtain; $\sum(S_i \Rightarrow P_i)' = \sum(S_i' + P_i)' = \sum(S_i P_i) = 0$. So the statement is equivalent to 'it is not true that there exists an S which is not_P'.

$$\textcircled{2} \text{ No } S \text{ are } P. \quad \prod(S_i \Rightarrow P_i') = \prod(S_i' + P_i) = \prod(S_i P_i)' = 1$$

Since the sentence 'No S are P' is the obverse of 'All S are non_P', we simply rewrite $\textcircled{1}$ replacing P by P'. Complementing both sides results in a \sum form as earlier: $\sum(S_i \Rightarrow P_i')' = \sum(S_i' + P_i)' = \sum(S_i P_i) = 0$ which says clearly that 'the statement that "there exists an S which is a P" is false'.

$$\textcircled{3} \text{ Some } S \text{ are } P. \quad \sum(S_i \Rightarrow P_i)' = \sum(S_i' + P_i)' = \sum(S_i P_i) = 1$$

Therefore $\textcircled{3}$ comes directly from the \sum -form of $\textcircled{2}$ in which it is asserted [to be true] that each of the terms is equal to zero or that none of them is true so that the infinite sum is zero.. Statement $\textcircled{3}$ says that at least one of them [and possibly all of them] is [are] equal to 1 so that the infinite sum adds to 1. By De Morgan's Laws we can write it in the \prod -form as: $\prod(S_i P_i)' = \prod(S_i' + P_i) = \prod(S_i \Rightarrow P_i) = 0$. The \prod -form seems unnatural for the existential quantifier since it says that 'it is false that all S are not_P or non_P'.

$$\textcircled{4} \text{ Some } S \text{ are not}_P. \quad \sum(S_i \Rightarrow P_i)' = \sum(S_i' + P_i)' = \sum(S_i P_i') = 1$$

This corresponds to 'There is some S which is not_P' and can be obtained from $\textcircled{3}$ simply by changing P to non_P. Similarly it is logically equivalent to the circumlocution expressed by: $\prod(S_i P_i')' = \prod(S_i' + P_i) = \prod(S_i \Rightarrow P_i) = 0$ or 'it is false that all S are P'.

It should be noted that if we adopt the convention that the representation in whatever form be equal to 1 [i.e. be true] then the \prod -form is natural form of representation for Universal/All/No statements; and \sum -form the natural form for the Some/Existential/Particular statements. Since complementing both sides doesn't change the truth value of the resulting equation, then negation is equivalent to complementing only one side and choosing the \prod -form or the \sum -form whichever is appropriate; that is satisfying the convention above. Therefore the quantifiers $\forall x$ and $\exists x$ should have no more significance than what was shown above except for the simple fact that if we used either the \sum -form or the \prod -form, we would either:

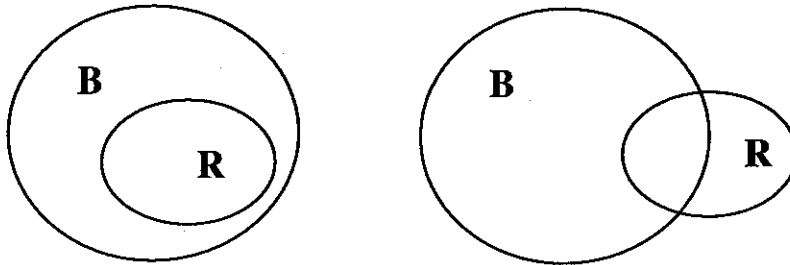
- (a) need to supply the summation limits or at least
- (b) be rudely reminded that we are somehow lacking something since we haven't put in the summation limits as would be required in any other branch of classical mathematics

It does then somehow seem to be significant that the notation $\forall x P(x)$ is chosen over the natural notation in which the results follow directly from the simple definitions of propositional logic and we might cast about for such reasons. In some cases, for example, quantifying over real numbers, {which are allegedly uncountable, whatever it means, see Hubey[1997]}, we'd be rudely reminded that we cannot perform such a

quantification. Since there is no such thing as uncountability and it's only a form of ω -inconsistency, it doesn't really matter but the use of the indeterminate universality embodied in \forall completely buries the issue. However, in using the Σ - Π forms we do gain something important [and which is certain to raise at least a philosophical eyebrow] ; that is, we can immediately and without any ambiguity show that if we have proven $\prod P_i$ then we have also proven $\Sigma P_i' = 0$, therefore [recalling that $\prod P_i$ really means $\prod P_i = 1$ and $\Sigma P_i' = 0$ is really the same thing, and also recalling that $\Sigma P_i'$ means $\Sigma P_i' = 1$] if we then prove $\Sigma P_i' = 1$, for sure there must be something wrong. Of course, this is routinely done in modern logic; it merely needs to be named i.e. ω -inconsistency. If we take the simple step of identifying $\prod P_i = 1$ with $\forall x P_x$, then just as simply, ω -inconsistency becomes a simple irrationalizable inconsistency. How is it possible that we can prove $P(1), P(2), P(3), \dots$, or $P_1 \cdot P_2 \cdot P_3 \dots$ in subscript or indexed notation, and still rationalize that we have not proven $\forall i P_i$ or equivalently $\prod P_i$? If proof of $P(1), P(2), P(3), \dots$ is not equal to proof of $\prod P_i$ and hence of $\forall i P_i$ we might as well give up Peano's axioms while we are at it. There can be only one real (pragmatic) reason why the above are not deemed equivalent, and that can only be the uncountability results and transfinites. It doesn't seem to be an accident that inductive reasoning opposes deductive reasoning. We seem to have met the fundamental incompatibility of inductive and deductive reasoning again, but the contradiction is illusory since it is inductive reasoning [the only real option in considering infinity] vs infinite totality in deductive methods. All the diagonal proofs involving infinity are based on the same fundamental problems involving infinite totalities, uncompleteable, never-ending proof methods and quite literally dubious results rationalized away to produce unconfirmable, unverifiable, and even unfalsifiable results. In order to even believe them one has to suspend disbelief. What kinds of deep truths in any branch of mathematics are artifacts of notation?

Sometimes, of course, we do not really prove either $\prod P_i = 1$ or $\Sigma P_i' = 0$. It's assumed to be already true. For example, if we start off with "Consider the reals in $[0,1]$.." it's not necessary to prove that all the reals in $[0,1]$ are really in $[0,1]$. Common sense says that "the set of all reals in $[0,1]$ " *really refers* to the fact that all the reals in $[0,1]$ really do have the property of being in $[0,1]$! It is too ridiculous to even contemplate finding a missing real from $[0,1]$. Where did it go, perhaps for a walk into the Twilight Zone? Conversely and circularly [or *consistently circularly*] we also assume that the "set of reals in $[0,1]$ " is comprised of all and only the reals in $[0,1]$. Then of course, using the technique that produces so many deep truths we can then prove that there's a real in $[0,1]$ that's not really in $[0,1]$! Then in order to hide the plain inconsistency of such a proof, we can make up a foggy concept called enumerability or listability and then claim via *reductio ad absurdum* that we have disproven our assumption. For this we should be thankful to the symbols \forall and \exists [and to Frege who seems to have brought up the artillery of medieval philosophy to provide service to Cantor], for they've done their job admirably. Unfortunately one thing refuses to go away; ω -inconsistency. Every instantiation of it can be given a name and labeled a deep theorem except where it's impossible to escape via official Orwellian naming ceremonies of definitions and trivial proofs all of which, of course, are the courtesy of infinite set theory. It's alluring to produce rigorous proofs of truths deeper than the deepest truths and just as impossible to give up the addiction to the proof methods that provide so many deep truths so rigorously. It becomes easier and easier to lift heavier and heavier rocks as time goes by. It seems we should take advice from nature: evolution cannot predict or determine its own future except via negative feedback

Consider the Euler diagrams for the ravens problem below.



The first diagram [left] could be a correct rendition of; *[All] Ravens are black*. We don't need anything more than propositional logic for this since it is really suppressed quantifier notation. Thus we can correctly render it as; $R \Rightarrow B$. We can also see from the diagram that we can also make the statement; "Some black-things are ravens" which unfortunately cannot be rendered in propositional logic with those symbols as we've used above. On the second diagram [right] we see that we can make the statement; "Some black (things) are ravens" as above. And it is just as correct that "Some ravens are black". Pretending that predicate logic symbolism had never been invented by Frege, we can actually see very clearly how we can make this simple language more expressive. The statement "All ravens are black" implicitly implies that we can index the ravens so that the statement can be written as

$$\textcircled{1} \quad \prod (R_i \Rightarrow B_i) = (R_1 \Rightarrow B_1)(R_2 \Rightarrow B_2) \dots (R_n \Rightarrow B_n) \dots = 1$$

The transcription essentially says; if the 1st object is a raven, then it is black, and if the 2nd object is a raven, then it is black, and ... We can also write it as;

$$\textcircled{2} \quad \prod (R_i \Rightarrow B_i) = (R_1 \Rightarrow B_1)(R_2 \Rightarrow B_2) \dots (R_n \Rightarrow B_n) \dots$$

where we've assumed the convention that the assertion that the [possibly] infinite product is true, similar to the suppressed summation notation with tensors. In other words, the assertion is already assumed to make the statement that it's true so that the equality can be left out. It's clear that we are leaving out the limits of \prod . It is therefore assumed that the product is from 1 to N where N is the number of ravens, however many of them might exist. We can even write it as;

$$\textcircled{3} \quad \prod (R_n \Rightarrow B) = (R_1 \Rightarrow B)(R_2 \Rightarrow B) \dots (R_n \Rightarrow B) \dots = 1$$

and interpret it as the; 1st raven is black, and the 2nd raven is black, and ... If we negate any of these equations, then we obtain

$$\textcircled{4} \quad \sum (R_i \Rightarrow B_i)' = (R_1 \Rightarrow B_1)' + (R_2 \Rightarrow B_2)' + \dots + (R_n \Rightarrow B_n)' \dots = 0$$

where the prime indicates negation. Since the implication $R \Rightarrow B$ is equivalent to $(R' + B)$ or $(RB)'$, if we write the original assertion as either

$$\textcircled{5} \quad \prod (R_i \Rightarrow B_i) = (R_1 \Rightarrow B_1)(R_2 \Rightarrow B_2) \dots (R_n \Rightarrow B_n) \dots = 1$$

A Simple Summary and Taxonomy

All - Some: The some (existential quantifier) is added in predicate logic. The all (universal quantifier) is already (hidden) in propositional logic. The addition of the existential quantifier \exists , then necessitates that we use an explicit symbol \forall for all which is really already implicitly part of propositional logic. The guiding principle seems to be modeled on De Morgan's Laws in the sense that the quantifiers themselves can be interpreted as primitives which like True and False do not behave like ordinary predicates since we have something like $(\forall P) \equiv (\forall' P') \equiv \neg(\forall' P') \equiv \neg(\exists \neg P)$. In another sense it duplicates De Morgan's laws since we can write [in prefix notation] \neg and $(x,y) = \text{or}(\neg x, \neg y)$.

Necessary - Possible: The idea of necessity like \forall is already hidden in propositional logic so modal logic seems to add another operator which obeys De Morgan-like laws as above since we have $(\diamond P) \equiv (\diamond' P') \equiv \neg(\square \neg P)$. It's obvious that this is trying to strengthen standard logic by plugging up the holes in the definition of implication by roundabout means. It's clear that if something is false, then it is necessarily false, and if something is true, then it's possibly true. How else can this be meaningful if truth was not tentative and falsity definite and absolute? Another interpretation of the combinations of the modal operators is simply as multivalued logics, where necessarily false/true are stronger and possibly true/false as partial [weaker] truth values. It may be possible to combine both the existential quantifier \exists and the unary possibility operator \diamond to form a ternary logic with a new connective \blacktriangleright which subsumes both modal logic and predicate logic.

Definite - Provisional: Apparently already part of propositional logic because of the definition of the implication/conditional. It's probably possible to explicitly add these operators to form a logic which may be better suited for a philosophy of science, than to try to describe the experimentation of science in propositional logic terms purely in terms of falsificationism. This idea also shows up in slightly different form; i.e. if a system says it's consistent, it maybe consistent or inconsistent, if however a system says that it is inconsistent, then for sure it is inconsistent, and we should believe it. The ideas of necessity and sufficiency which are normally used with the definition of (material) implication seem to another example ideas related to the definite-provisional dichotomy in which there is a fundamental asymmetry or bias in logic.

- \blacktriangleright Whenever A is true, then B is true.
- \blacktriangleright A can be true only if B is true.
- \blacktriangleright A necessary condition for A to be true is that B is true.
- \blacktriangleright It is necessary that B be true in order for A to be true.

When a statement A implies a statement B, we say that B is a necessary condition for A. This is the real definition of the implication, and also the definition of necessity. We can then define sufficiency, in terms of necessity and the implication; thus A implies B is also equivalent to saying that A is a sufficient condition for B. This is equivalent to saying that the truth of A is sufficient for us to conclude the truth of B. This statement of sufficiency can also be stated in different ways;

- \blacktriangleright A sufficient condition for B to be true is that A is true.
- \blacktriangleright For B to be true, it is sufficient that A is true.

$$\textcircled{6} \quad \Pi(R'_i + B_i) = (R'_1 + B_1)(R'_2 + B_2) \dots (R'_n + B_n) \dots = 1$$

$$\textcircled{7} \quad \Pi(R_i B'_i) = (R_1 B'_1)(R_2 B'_2) \dots (R_n B'_n) \dots = 1$$

then the negation is most naturally written as

$$\textcircled{8} \quad \Sigma(R_i B'_i) = (R_1 B'_1) + (R_2 B'_2) + \dots + (R_n B'_n) \dots = 0$$

Since each term says that there is no object which is a raven and not black, equation $\textcircled{8}$ is still equivalent to the original assertion that "ravens are black". However, equation $\textcircled{8}$ is written in a form suitable for an intuitive comprehension of the existential assertion that "there exists a raven which is not black". It is simply

$$\textcircled{9} \quad \Sigma(R_i B'_i) = (R_1 B'_1) + (R_2 B'_2) + \dots + (R_n B'_n) \dots = 1$$

It asserts that at least one of the terms is not zero, and hence there is at least one raven which is not black since we should have $(R_j B'_j) = 1$ or $(R'_j + B_j) = 0$ for some j . Obviously this condition would force eq. $\textcircled{8}$ and hence the original assertion to be equal to zero and hence false. The only advantage of the notation $\forall x$ and $\exists x$ would seem to be in hiding the fact that the indexing scheme is in fact being employed, and in making it unnecessary to produce explicit limits for the product and sum. It's obviously advantageous to be able to quantify over uncountable sets [if such things exist] by simply obfuscating the meaning of \forall and \exists . It's very easy to see where the ω -inconsistency comes from as soon as one looks at the quantifications as implicitly using Π and Σ .

3. Why is the Liar Paradox a Paradox ?

Static model of truth does seem to have its limits as does bivalency. There seems to be quite a different way in which truth values behave which seems to indicate that it's simply not true that truth or falsity are undefined primitives in the way it's used in logic. The primitives of logic are T and F, and it shouldn't seem too strange that they could/should be treated specially in logic. One case of this shows up when the intuitionists use the falsum \perp , in special ways in their axioms. A related problem crops up in logic, in that it is not normally used as just another algebra since the actual assignments of values of variables is not done but rather the default is used in that variables are normally thought of as being true (i.e. positive logic). However, we can explicitly assign values, and show the steps. In this case, it can be shown that a particular method of assigning values must be used and which will be shown to be consistent with the standard practice but shows that the liar paradox arises out of a mispractice of explicit value assignment. For example, suppose we write *that P is true* or *that P* as $P \Leftrightarrow T$ then we have (using algebraic manipulation)

$$\textcircled{6} \quad (P \Rightarrow T)(T \Rightarrow P) = (\neg P + T)(F + P) = PT = P$$

where the equals sign, =, is used as in algebraic calculations instead of the biconditional in order to avoid confusion. It is preferable to let the biconditional denote the equivalence in the assertions the truth of which we are attempting to ascertain, and to use the = in showing the equivalences using the rules of Boolean algebra. One may then interpret the implication as *possession of a property*, and since only sentences/propositions may possess the property of being True/False, the interpretation fits as well as any other interpretation. Otherwise we have to go through verbal contortions to try to explain what $(P \Rightarrow T)$ could

mean. As it is $(T \Rightarrow P)$ is difficult enough to interpret. If we attempt it via analogy to $F \Rightarrow P$, then it doesn't mean much except that 'from a true statement one cannot derive anything!' which is not really something anyone wants. Similarly $P \Leftrightarrow F$ computes to P' as it should.

$$7) \quad (P \Rightarrow F)(F \Rightarrow P) = (\neg P + F)(T + P) = \neg PT = \neg P$$

We can even construe it as 'It is not true that P is true' whence

$$8) \quad \neg(P \Leftrightarrow T) = \neg[(P \Rightarrow T)(T \Rightarrow P)] = \neg[(\neg P + T)(F + P)] = \neg[PT] = \neg P$$

One is tempted at this point to rename the equivalence as co-extensivity since the interpretation seems to imply it. Therefore, if any animal that has a heart has kidneys, and any animal that has kidneys has a heart, we can write it as $(H \Rightarrow K)(K \Rightarrow H)$ since it naturally reduces (algebraically) to $HK + H'K'$ which is equivalence (co-extensivity). Only in the case in which the two variables are the same (i.e. $A \Rightarrow A$) does it reduce to a tautology via the Law of the Excluded Middle. We should note that even if we construe the \Rightarrow as 'possessing a property' the interpretation still seems to be valid, since then $S \Rightarrow W$ could mean that 'Snow possesses the property of being white'. However the co-extensivity of kidneys and heart (as written above) is a shorthand since we neglected to write anything about animals. It should have been written as; $A \cdot [(A \Rightarrow K) \Rightarrow (A \Rightarrow H)][(A \Rightarrow H) \Rightarrow (A \Rightarrow K)]$ which reduces to $A(HK + HK)$ as we expect since everything is explicit. Even here we have not included the explicit assignment of the claims that the statement is true. The double-duty of the implication sign is evident here despite attempts to give it a single meaning since the middle \Rightarrow in the first part (i.e. $(A \Rightarrow K) \Rightarrow (A \Rightarrow H)$) is more difficult to construe as possessing a property. The first part can be interpreted as 'A possesses K' and the second as 'A possesses H', so the whole thing should/could read something like (with strained English) 'A that possesses H, possesses $A \Rightarrow K$; in other words A also possesses the property of *A-possessing-H*'.

Continuing the thread of the implicit existence of the assignment to truth, we see that if we depict it (*that P or P*) as $P \Leftrightarrow (P \Rightarrow T)$ not only does it still work to the same since "P" is equivalent to "P is true" as demonstrated below, but it is the correct way the statement should be depicted, instead of the shortcut as above and instead of the even shorter cut 'that P' as is usually done. From the definition of the biconditional

$$9) \quad [P \Rightarrow (P \Rightarrow T)][(P \Rightarrow T) \Rightarrow P] = [P' + (P' + T)][(P' + T)' + P] = \\ = [P' + T][(PF) + P] = [T][F + P] = F + P = P$$

Then $P \Leftrightarrow (P \Rightarrow T)$ seems as it is really the statement 'P is equivalent/co-extensive to/with the statement that P possesses the truth-value of True'. Therefore we have the statement that *Snow is white* is equivalent to: *Snow is white is true* or *It is true that 'Snow is white'* or *The statement that 'Snow is white' is coextensive with the statement that 'Snow is white' possesses the property of being True*. It is just as easy to show the equivalence by substituting the expression $S \Rightarrow W$ into the above for P. It is just as easy to show that *that P* is equivalent to *that not-P is not-False* just as one would expect from their meanings; that is $P \Leftrightarrow [P' \Rightarrow F']$ as shown below where T is substituted for place of F'

$$10) \quad [P \Leftrightarrow P' \Rightarrow F'] = [P \Rightarrow (P' \Rightarrow T)][(P' \Rightarrow T) \Rightarrow P] = \\ = [P' + P + T][(P + T)' + P] = [T + T][P'F + P] = [T][F + P] = TF + TP = P$$

So far we do not seem to have reasons to prefer either of the forms above to each other, nor do we have any

reasons to use these forms instead of the much shorter and standard P or $\text{not } P$. However these are not correct since the Truth-teller paradox nor the Liar paradox get reasonable interpretations as shown below. These operations don't work for any other property or predicate except for the primitive truth values [T and F treated as primitive values or properties] so that the definitions of truth and falsity are already buried in propositional logic in the way the word "is" is interpreted. In a sense, the semantics are already a part of the coding of the syntax [for more along these lines see Hubey[1997]]. In terms of the above discussion we can examine the Liar's Paradox and the Truth-teller Paradox differently than is normally done. Since the assertion *that P* is equivalent to *that P is true*, the self-referential statements are not merely asserting anything along the lines of *that P* but are asserting that the truth value of the statement in question be assigned to the statement itself. Therefore the statement P that asserts *This sentence is false* can, then, be written as asserting $P \Leftrightarrow (P \Leftrightarrow F)$ since the second factor/term is equivalent to asserting that *P is false*, but this is the truth value that the sentence asserts should be assigned to the sentence, i.e. itself. Obviously, we then obtain $P \Leftrightarrow P'$ which is $(P \Rightarrow P') \wedge (P' \Rightarrow P)$ and definitely a contradiction. The problem is one of not being able to assign a fixed truth value to the sentence but this is not the only way a paradox can arise. If anything, a sentence that asserts it's false is like a system that asserts that it is not consistent; we have to believe that there is something wrong!

The real problem or paradox, is if the system asserts that it is consistent or that the statement is true. The depiction of the Truth-teller Paradox obtains when we write the sentence "*This sentence is true*" as $(P \Rightarrow P)(P' \Rightarrow P')$ [i.e. true if true and false if false] which, of course, is a tautology and which could pass as two applications of the Law of the Identity. But this is the wrong result, since we don't know that the sentence is true. This can easily be construed as yet another example of the intuitionistic case against the Law of the Excluded Middle since the statement "*This sentence is true*" is true if it is true and false if it isn't [or maybe false or true if it's not true]. Using similar reasoning to above we should then code or depict this sentence as $P \Leftrightarrow (P \Leftrightarrow T)$ which is equivalent to $P \Leftrightarrow P$ which by definition is $(P \Rightarrow P)(P \Rightarrow P)$ and reduces to the *Law of Identity*, $P \Rightarrow P$. It's true that everything is itself but the Truth-teller computes to $P + P'$ which becomes a tautology via the Law of the Excluded Middle. However, it is most certainly true that a sentence which asserts its verity is about as reliable as a liar insisting that he's telling the truth or a system producing a proof that it is consistent. The Truth-teller sentence should compute to P ; so it's true if it's true and false if it's false. These paradoxes and others like them which pop up in metalogic share the property that, like the future contingent, the act of attempting to assign truth values to the proposition, causes its truth value to change since its truth value is dependent on the truth value that we attempt to assign. Therefore the coding of the Liar and Truth-teller as above is not correct, since we expect the coding to be consistent and correct.

So taking a cue from the standard conditional and the specific treatment meted out above to the Liar Paradox, we might attempt to write it as $[P \Rightarrow P][P' \Rightarrow (P + P')]$ but it's still a tautology and not a correct rendition. The sentence should really assert itself and nothing more. We can however construe the sentence as $(P \Rightarrow P)(P' \Rightarrow P')(P \Leftrightarrow T)$ [true if true, false if false but *asserted to be true*] or as $P \Leftrightarrow (P \Rightarrow P)(P' \Rightarrow P')$ [making use of the artifice of assigning the truth value of the sentence to itself as in the case of the liar sentence]. In both cases it computes to P , as it should. It should be noted that the liar's sentence, if interpreted this way, still computes to false since we would then have to write it as $(P \Rightarrow P')(P' \Rightarrow P)(P \Leftrightarrow F)$ or $P \Leftrightarrow (P \Rightarrow P')(P' \Rightarrow P)$ which compute again to a contradiction. The reasoning does get more convoluted but it's still a lot simpler (to the author) than any of the other efforts such as not assigning truth values, creating truth value gaps, or the diagonal solution of pushing the problem to another stage. The only thing that has been done is to use the existing machinery of logic to rewrite statements in such a way as to show that what we have been using is short hand or simplification which works most of the time except in pathological cases. Although it seems that at least some part of the problem may have been solved (of course, in all likelihood not to

everyone's or maybe to anyone's satisfaction) there are more problems simply because it's in the nature of the implication/conditional. We immediately obtain an apparent snag if we rewrite the original assertion above, (i.e. eq(9) $P \leftrightarrow [\neg P \Rightarrow \neg F]$), as $P \leftrightarrow (F \Rightarrow P)$ using the contrapositive since we know that $(F \Rightarrow P)$ makes any proposition P automatically true. However, we now have reasonably good justification for insisting that we should instead insist that the above form [i.e. $P \leftrightarrow (F \Rightarrow P)$] is one of the correct forms for asserting P and that $(F \Rightarrow P)$ should not form any part of propositional logic since its interpretation immediately forces it to make statements not about verity/falsity of propositions but of the whole system of propositional logic. It seems that making statements about logic belongs to metalogic. But in the above form it's correct since $F \Rightarrow P$ computes to T and hence the above becomes $P \leftrightarrow T$, which is really the assertion that P is true, as in eq.(6). Neither the Liar nor the Truth-teller is a paradox, but the underlying apparatus was being hidden because of the tendency to use logic in more natural-language like settings instead of like other algebras in which assignments of values, and manipulations are much more explicit, however there is yet another intriguing way in which the Liar-Paradox-like concepts are important. We can summarize

TABLE II

	<u>Coding</u>	<u>Interpretation</u>	<u>Evaluates to</u>	<u>Comments</u>	
①	$P \leftrightarrow T$	P is True	P	Correct	
②	$P \leftrightarrow F$	P is False	P'	Correct	
③	$P \Rightarrow T$	P is True	T	Error	Truth-teller Paradox
④	$P \Rightarrow F$	P is False	P'	Correct	
⑤	$P \leftrightarrow (P \leftrightarrow T)$	This sentence is True	$P \leftrightarrow P$ (i.e. T)	Error ?	Truth-teller Paradox
⑥	$P \leftrightarrow (P \leftrightarrow F)$	This sentence is False	$P \leftrightarrow P'$ (i.e. F)	Correct	Liar Paradox
⑦	$P \Rightarrow (P \leftrightarrow T)$	This sentence is True	$P \Rightarrow P$ (i.e. T)	Error	Truth-teller Paradox
⑧	$P \Rightarrow (P \leftrightarrow F)$	This sentence is False	$P \Rightarrow P'$ (i.e. P')	Error	
⑨	$P \leftrightarrow (P \Rightarrow T)$	This sentence is True	$P \leftrightarrow P$ (i.e. T)	Error	Truth-teller Paradox
⑩	$P \leftrightarrow (P \Rightarrow F)$	This sentence is False	$P \leftrightarrow P'$ (i.e. F)	Correct?	Liar Paradox
⑪	$P \Rightarrow (P \Rightarrow T)$	This sentence is True	T	Error	Truth-teller Paradox
⑫	$P \Rightarrow (P \Rightarrow F)$	This sentence is False	$P \Rightarrow P' = P'$	Error	Liar Error

From the above table, we can see that ①, ②, ⑤, and ⑥ seem to be the correct versions with some patching up. ① and ② are definitely correct, since this formulation becomes exactly the shortcut we use when we assert 'P is true' or 'P is false' (implying the possession of a property as explained earlier). However, later on (⑤, and ⑥) we attempt to use \leftrightarrow in meaning close to assignment. The operator \leftrightarrow should be used for coextensivity, since in general it means that since Quine's example (i.e. All animals that have kidneys have a heart, and all animals that have a heart have kidneys). Indeed if we have $A \leftrightarrow B$ it computes obviously to $AB + A'B'$. However in the special case in which A is the same as B (note we are not using any signs like $=$, $:=$, \Rightarrow , or \Leftrightarrow) then we can make the special rule that we have $A \leftrightarrow A$ which is really $(A \Rightarrow A)(A \Rightarrow)$ hence really $A \Rightarrow A$ which naturally is either the Law of Identity or true via the Law of the Excluded Middle (LEM), and which gives the wrong results above in the Truth-teller Paradox (i.e. ⑤). If anything, this buttresses the viewpoint of the intuitionists in denying unrestrained application of the LEM. The assertion that the sentence is true computes to $P \leftrightarrow P$ (and we should not go further and compute it to be T since it only says that the statement is equivalent to itself. Similarly ⑦ can't be correct because then ⑧ which is composed via analogy gives the wrong result since it also asserts that what was asserted is true (which is that it is false). Via a similar analogy although ⑩ gives the correct final result, a part of it $P \Rightarrow F$ gives the wrong result when we write it as $P \Rightarrow T$ as in ③, so the best overall solution seems to be as described above with the slight reinterpretation of the operators as explained above.

4. Parametric Representation--Return of the Liar

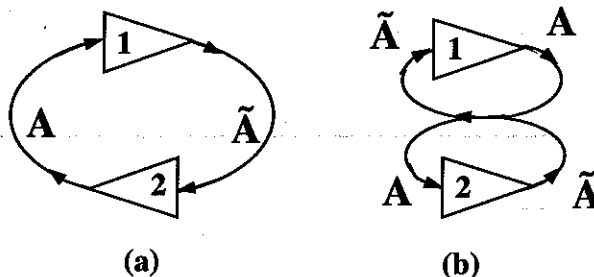
The liar paradox brings up two interesting observations. One of them has to do with representing logical variables as parametric equations, and the other about the self-reference which could be in inferencing but immediately noticeable. We know that in algebra the equation;

$$11) \quad P^2 + R^2 = 1$$

is the equation of the unit circle. It shows the relationships of the two variables for all time; time is implicit. However the behavior of these two variables in time is given by

$$12) \quad R(t) = \cos(t) \quad \text{and} \quad P(t) = \sin(t)$$

The second description is a parametric description, it's the solution of the well-known harmonic oscillator. In the first description time is implicit; it shows the relationships of x and y together (and) for all time. Of course, we can put time explicitly into the equation of the circle. Similarly the chain of iteration (reasoning/deduction) that produces the oscillation is best understood as a parametric description. We collapse this oscillation into a frozen time by ANDing them together similar to producing the equation of the circle. Logic is richer than we suspect and we keep getting solutions that we don't want or know how to interpret. More correctly some of the solutions of Boolean Algebra or Logic are not interpretable in the sense in which we want and we resort to ad hocery. We keep pretending that the act of inferencing has no bearing on the truth of the final conclusion but the Liar paradoxes say it's not so. We cannot assign a fixed value because the value depends on the value we assign. Logic has hit its limits. There are problems for which



**Parametric Representation
of the Liar Paradox**

we get oscillatory solutions and these are self-referential, or viciously circular. We can produce deep truths from these by ignoring other possibilities and other perspectives. The only time we accept them seems to be if they are beyond our intuitive capabilities to refute, hence once again, *true until proven beyond a shadow of doubt that they are false*. Imagination is a wonderful thing; all the great scientists say so. But something seems to have been overlooked by some of them; many people with great imaginations are not scientists at all, not even science fiction writers. The circular double inverter above is in some ways the standard Liar Paradox representation. The reasoning is simple, the sentence "This sentence is False." actually says two things; (a) If assume the sentence is true, then it's false, (b) if we assume it's false, then it's true, which can easily be written as $(A \Rightarrow A')$ and $(A' \Rightarrow A)$ which quite naturally computes to False since we have (explicitly using the assignment) $(A \Rightarrow A') (A' \Rightarrow A) = (A' + A')(A + A) = A'A = F$. The figure above, of course, is essentially the circuit for the flip-flop; only the OR gates are missing but they are not really necessary except for making the actual circuit. The circuit above is stable.

5. The Crocodile Paradox and Limits of Truth

The possible solutions of the Crocodile paradoxes depend [as always] on how we interpret the agreement between the mother and the crocodile [please see appendix]. It seems that they both agree that if the prediction is correct, then the baby will be returned. In addition, the croc also says the if the prediction is not correct then he'll devour the baby, that is, the baby will not be returned, so then we also have

$$13) \quad P \Rightarrow R \quad \text{and} \quad P' \Rightarrow R'$$

Thus no matter what the end result, we already have according to the crocodile

$$14) \quad (P' \Rightarrow R')(P \Rightarrow R) (\dots) \Rightarrow [R? \text{ or } R' ?]$$

The left side computes to $PR + P'R'$ so that if we multiply by $P+R$ we'll obtain PR and if we multiply by $P'+R'$ we'll obtain $P'R'$. Complementing the former yields $P'+R'$ while complementing the latter yields $P+R$. The resolution is symmetric so that the former added to R is a tautology as is the latter added to R' . Thus in order to create an impass we need to create one [or more] inferences that will result in this impass. The crocodile contends that if he returns the baby then the mother has not made the correct prediction so he claims $R \Rightarrow P'$ which computes to $R'+P'$ exactly as needed so that the crocodile's position is:

$$15) \quad \begin{array}{l} (P' \Rightarrow R')(P \Rightarrow R)(R \Rightarrow P') \Rightarrow R' \\ (P'R')' + R' \\ P + R + R' \quad \text{Tautology} \end{array}$$

Meanwhile the mother contends that if the crocodile does not return the baby, then the prediction is true i.e. $R' \Rightarrow P$ just as required since it computes to $R+P$

$$16) \quad \begin{array}{l} (P' \Rightarrow R')(P \Rightarrow R)(R' \Rightarrow P) \Rightarrow R \\ (PR)' + R \\ P' + R' + R \quad \text{Tautology} \end{array}$$

One could read more or less into the paradox and produce different solutions but the real interest in the paradox is exactly because two differing [indeed contradictory] conclusions can be produced simply because the action of returning or not returning affects the assignment of truth value to the prediction and the action to be taken i.e. *to return or not to return* depends on the prediction. In this case, the simple act of decreeing the empty set to be a subset of every set doesn't work. The singularity persists. Just as *inside the bag* is not equivalent to the *not outside the bag*, returning and not returning cause havoc with truth values. If we want to explicitly show the algebra as in the assignments of truth values to variables instead of using mental defaults, what this boils down to is the attempt to simultaneously make truth assignments of the form (where the assignment operator $:=$ is explicitly used) :

$$17a) \quad P := R' \quad \text{or} \quad P = R'$$

$$17b) \quad R := P \quad \text{or} \quad R = P$$

We've used the assignment sign/symbol since the prediction is true if the baby is not returned, and false if

the baby is returned. At the same time, the baby is supposed to be returned if the prediction is true and not returned if the prediction is false. If we consider this to be an algebra of sorts we'd simply go through the motions of solving the simultaneous set of equations, as in $R=P=R'$ which is a contradiction. Since we try to produce inferences sequentially, this is also attempting to assign a fixed value in which case we'd simply find that we'd obtain oscillations of the form R,R',R,R',R,\dots or R',R,R',R,\dots depending on where we start. It's the same old unstable fixed point problem. It can be written simply as a vector equation as shown below.

$$18) \quad \begin{vmatrix} P \\ R \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} P \\ R \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} R \\ P \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} R \\ P \end{vmatrix}$$

Not much more can be said except that this is an obvious analog of the vector differential equation which gives us the famous harmonic oscillator as shown below. It would seem that we should seek the solution, if there is one, in more generalized systems of which bivalent logic is a subset, for example continuous valued logics. Although truths of logic are supposed to be constant and timeless truths [unless otherwise specified] it's strange that logic doesn't allow for simultaneous representations such as in vector, or matrix algebra or differential equations in which time is made explicit.

$$19) \quad \frac{d}{dt} \begin{vmatrix} R \\ P \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} R \\ P \end{vmatrix}$$

It should be noted that the vector equation above arises out of similar apparently contradictory behavior. If we have $R'(t) = P(t)$ and $P'(t) = -R(t)$, [where the prime indicates a time derivative] we simply differentiate the first to derive the harmonic oscillator, i.e.

$$20a) \quad R''(t) = P'(t) = -R(t) \quad \text{or}$$

$$20b) \quad R''(t) + R(t) = 0$$

This equation, of course, has the solutions as in Eqs.(11) and (12), and would be perfectly reasonable in an infinite valued logic in which the values were restricted to $[-1,+1]$ instead of the usual $[0,1]$. Indeed there are reasons why $[-1,+1]$ is a better choice; we can always map it onto $[0,1]$. This flip-flopping of values in logic and its analogues in other fields and still unanswered anomalies can be seen in the works of Spencer-Brown in which the standard equation whose solution (resolution ?) gave rise to imaginary numbers [i.e. $R^2 = -1$] is written as

$$21) \quad R = -1/R$$

And it can be seen that if 1 or -1 is substituted for R we obtain a flip-flopping $+1,-1,+1,-1,\dots$. If we substitute values other than 1 or -1 the equation still flip-flops between two values. If we plotted x_n vs x_{n+1} , we would find that the points fall on hyperbolas and flip between the initial value x_0 and $-1/x_0$. Something along the same lines can be seen if we follow this chain of reasoning:

$$22a) \quad 1/3 + 1/9 + 1/27 + \dots = 1/2$$

$$\begin{aligned}
 22b) \quad & (-1)^{1/2} \equiv i \\
 22c) \quad & \therefore i = (-1)^{1/3 + 1/9 + 1/27 + \dots} = (-1)^{1/3} (-1)^{1/9} (-1)^{1/27} = \\
 & = (-1)(-1)(-1)\dots = +1 \text{ or } -1
 \end{aligned}$$

It does seem as if Spencer-Brown's [1973] thoughts about the oscillation of the truth values indicating some connection with imaginary/complex values might have some usefulness. Perhaps mathematical logic should go back to algebraic methods, among which one should expect to find infinite valued logics and even-other logic-like systems. The no man's zone of verbiage between science and speculation has always been inhabited by a peculiar brand of humans whose imaginations seem to outweigh their knowledge. As science progresses and casts its net wider and wider, the inhabitants of these transition zones will always get pushed further and further toward the fringes. The results above are tantalizingly close to pointing to some deep relationship between logic, complex algebra, and probability theory as seen in quantum mechanics in which the wave function is complex and "meaningless" (at least within the present framework of mathematics) but the modulus of the wave function is a probability density. One can find justification for producing a kind of probabilistic explanation related to confirmation theory from set theory and logic under appropriate approximations but the set theoretic standard of formalism blocks its manifestation as anything other than intuition. However it is possible to produce mathematical structures which do meet at least some of the conditions of "rigour" imposed by the prevalent mode of set-theoretic thinking. The real question is why we insist that we must be able to assign a specific set of values to a set of Boolean variables which do not oscillate. Are diagonal proofs problems of systems as powerful as arithmetic or is it a part of Boolean Algebra?

To peek further into this process of oscillation and to find its relationships we should be able to represent the complementation operator to something which is more conducive to algebraic manipulation. The most obvious solution is to use only the values [+1,-1] instead of [0,1] since then multiplying by unary minus complements both True (i.e. 1) and False (i.e. -1). Then eq. (19) makes perfect sense as fuzzy or infinite-valued logic. For the still bivalent, discrete logic, we can then rewrite eq. (18) as

$$23) \quad \mathbf{v} = H_1 \mathbf{v} \quad \text{or} \quad \mathbf{v} = H_2 \mathbf{v}$$

where the bold letters denote vectors, so $\mathbf{v} = \begin{vmatrix} P \\ Q \end{vmatrix}$ $H_1 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$ $H_2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$

We should note that there's an extra benefit from this specific choice of truth values in that we can now explicitly show 'truth value gaps' if we so desire, by simply multiplying it with zero. Since the only way we can achieve this oscillation is not via H_1 , we can see that we need more matrices. Defining

$$24) \quad H_3 = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \quad H_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

we can immediately show

$$\begin{aligned}
 25a) \quad & H_3 = H_1^2 = H_2^2, \quad H_1 H_3 = H_3 H_1 = H_2, \quad H_3 H_2 = H_2 H_3 = H_1 \\
 25b) \quad & H_1 H_2 = H_2 H_1 = H_0, \quad \text{and} \quad H_0 H_k = H_k H_0 = H_k \quad \text{where } k=0,1,2,3
 \end{aligned}$$

We should note that multiplication is now of two different kinds. In one case, in which the matrix multiplies

the vector we have $(-1)(+1)=-1$ and $(-1)(-1)=1$ since this multiplication is really complementing the vector components which are Boolean variables. We cannot in general have $(-1)(-1)=1$ since it would mean $FF=T$. Summarizing this in a table as given below, we can see that the H_k comprise a group with H_0 as the identity.

26)

		H_m			
		H_0	H_1	H_2	H_3
H_n	H_0	H_0	H_1	H_2	H_3
	H_1	H_1	H_3	H_0	H_2
	H_2	H_2	H_0	H_3	H_1
	H_3	H_3	H_2	H_1	H_0

Multiplication table
for $H_m H_n$

A more interesting problem is finding the eigenvalues of the H matrices. In all cases we obtain $|\lambda|=1$. For H_1 and H_2 , we have $\lambda_{1,2} = \pm i$, for H_3 , $\lambda_3 = -1$; and naturally for H_0 which is idempotent (and the identity of the group) we have $\lambda_0 = 1$. Strangely enough the eigenvalues also form a group under multiplication. They form a group also even including zero under multiplication. It's not clear exactly how these can be interpreted. The zero can certainly be used as a "truth-value" gap, and so can the imaginary values if we consider bivalent logic using $\{-1,+1\}$. If we use continuous valued logic, then 0 still makes sense and it can even be used in ternary logic. Both H_1 and H_2 serve to create the oscillation and because there are no real eigenvalues corresponding to these, we have an algebraic proof that there is no real solution to the Crocodile Paradox. We have something similar to the [in]famous problem of the meaning of the wave function ψ in Schroedinger's equation. The modulus $|\psi^2|$ is interpreted as a probability density but ψ is complex. We seem to have a similar problem with the H-operators above, and their eigenvalues. Perhaps a meaning can be attached to them, if this formalism can be used to model some real world problem. Research is on the way toward achieving this goal. All of the results shown in this paper point to an oscillation, a refusal of the system to converge to a stable solution, and an impossibility of assigning either a True or a False value to the statements in question. In addition to the possibilities offered here, there are also possibilities of discussing the problems in terms of fuzzy logics [Hubey,1996b], and even special classes of fuzzy logics in which the analogy to probability theory and set theory is plain since the $AND(x,y)$ and $OR(x,y)$ functions can be written as

$$27a) \quad AND(x,y) = t(x)t(y)$$

$$27b) \quad OR(x,y) = t(x) + t(y) - t(x)t(y)$$

where $t(x)$ is a truth-valuation function. The simplest such case is the product-sum fuzzy logic in which $t(x)=x$ thus eqs.(27) are

$$28a) \quad AND(x,y) = xy$$

$$28b) \quad OR(x,y) = x + y - xy$$

In the above case and the ones developed in Hubey[1997] the complement is not only involutive but also satisfies $c(x) = c(t(x)) = 1 - x = t(1-x)$. Therefore it is shown that the union/AND and intersection/OR satisfy De Morgan's Laws [Klir & Yuan, 1995] and hence are duals of each other. In that sense they can be treated as transition between logic and probability theory.

If we wanted to create a logic which used only $\{-1,0,+1\}$ where False were -1 , True 1 , and 0 as a truth-value gap, there are many ways in which we can define $OR(A,B)$, and $AND(A,B)$. For example using $U(x)$ for the Heaviside Unit Step Function which is 1 for $x > 0$, and 0 for $x < 0$

$$29a) \quad OR(A,B) = U(-A) \cdot B^A + U(-B) \cdot A^B + U(A \cdot B)$$

$$29b) \quad OR(A,B) = U(A) \cdot A^B + U(B) \cdot B^A - U(A \cdot B)$$

$$29c) \quad OR(A,B) = U(A) + U(B) - U(A \cdot B)U(A)U(B)$$

$$30a) \quad AND(A,B) = U(A) \cdot B^A + U(B) \cdot A^B - U(A \cdot B)$$

$$30b) \quad AND(A,B) = -U(-A) \cdot A^B - U(-B) \cdot B^A + U(A \cdot B)$$

$$30c) \quad AND(A,B) = U(A \cdot B) \cdot U(A) \cdot U(B)$$

Some of these systems evaluate to imaginary or complex values for some values of A , and B , and can be intuitively useful in pointing out that there are no (real) solutions. If however we wanted to be able to produce results as in classical logic, probably the easiest would be to first define OR and AND for $\{0,1\}$ and then define new ones in terms of these for $\{-1,+1\}$. For example, define the classical AND and OR as

$$31a) \quad AND_c(A,B) = U(A-1/2) \cdot U(B-1/2)$$

$$31b) \quad OR_c(A,B) = U(A-1/2) + U(B-1/2) - U(A-1/2) \cdot U(B-1/2)$$

so that the proper values are obtained at the boundaries, and then define

$$32a) \quad AND(A,B) = 2 \cdot AND_c(A,B) - 1$$

$$32b) \quad OR(A,B) = 2 \cdot OR_c(A,B) - 1$$

We can see that eqs. (31) are in the same form as eqs (27) but unlike that of eqs. (28) do not possess values in $[0,1]$ but only have two values in $\{0,1\}$. Even if we substituted values for A or B in $[0,1]$ in eqs. (31) the results would be in $\{0,1\}$ because of the definition of the Heaviside function, and furthermore the bias of $1/2$ guarantees that values of $A, B > 1/2$ produce the same results as $A=B=1$, and similarly for $A, B < 1/2$ the results are as if $A=B=0$. However even in this case, as in the eqs. (28) there is a value of A , and B (i.e. $A=B=1/2$) which is a singularity. In the case of the use of the Heaviside function, that singularity is sometimes arbitrarily shifted to one side, and sometimes in the definition $H(0)$ is defined to be $1/2$. In the case of the oscillation/iteration using a logic such as (28), the results converge to truth values of $1/2$, which is neither True nor False. The use of functions such as these shifts the burden of doing logic over to the functions used and reminds us that instead of remembering simple rules such as $1+1=1$ in Boolean Algebra, we are now burdened with using iterations with highly nonlinear functions, and we can expect chaos in the results.

Appendix A: The Crocodile Paradox

Liar paradoxes have a long history and are shown in many books on logic. There are whole books dedicated to the explanations of these paradoxes; such as Martin[1970,1984] and nonstandard set-theoretic dispositions of these due to Barwise and Etchemendy[1987] using Aczel's non-wellfounded sets. However, the treatment of these paradoxes in this section will use standard bivalent propositional logic and simple set theory. The point of view taken in this exposition is like that of Curry, and Parsons and unlike that of Ramsey; semantic paradoxes and set theoretic/logical paradoxes are considered to be of the same type. Similarly it's not necessary to either take the Russell-Tarski approach of multilevels or the necessity of using more than two values although it's very convenient to use multivalent logics. An excellent discussion of the crocodile paradox from which many of the deep truths of logic as well as the olden and modern paradoxes such as the simple Liar Paradox can be found in Falletta [1990] [italics mine]:

A crocodile seized a human baby who had been playing on the banks of the Nile. The implored the crocodile to return her child. "Well" said the crocodile, "if you can predict accurately what I will do, I will return the child. However, if you guess wrong, I will eat it for my lunch."

"Oh, you will devour my baby!" cried the distraught mother.

"Now," said the wily crocodile, "I cannot return your baby, for if I do return it, I shall make you speak falsely and I warned you that if you spoke falsely I would devour it."

"Quite the contrary," said the clever mother. "You cannot devour my baby, for if you do devour it, you will make me speak the truth and you promised me that if I spoke truly, you would return my baby. I know you are an honorable crocodile and one who will keep his word."

.. The oldest treatment of this dilemma is traced to Diogenes Laertius, the ancient Greek biographer who lived in the third century A.D., although other references indicate that it may go back to the Sophist philosophers of the fifth century B.C. The dilemma is related, in its history and its structure, to the lawyer's paradox; it is really an elaborate variation of the liar paradox.

...The crocodile's argument can be expressed as follows:

If the mother accurately predicts what I will do, then I will return the baby.

If I return the baby, then the mother has not accurately predicted what I will do.

Therefore, I will not return the baby.

The mother's view can be expressed in the following form:

If I accurately predict what the crocodile will do, then my baby will be returned to me.

If my baby is not returned (that is, devoured), then I predicted accurately what the crocodile will do.

Therefore, the baby will be returned to me.

There are plagiarisms or variations of this theme in many places. The simplest one is the Liar Paradox. There is one in which the author sticks his hand into a hole in some marble and says "I will not be able to pull my hand out." Since the legend says that anyone who utters a falsehood with his hand in the hole will not be able to pull his hand out, and since the author pulled his hand out then he must have told the truth,

but then he pulls his hand out anyway. Another author has a machine that only prints true sentences but cannot print the sentence "This sentence is not printable" and accordingly it's an example of a true sentence that is not provable [not printable] since the printing of the sentence corresponds to the machine proving it. Of course, then there are the Gödelian theorems; that there exist unprovable truths. But no one has ever produced a true statement of arithmetic that is not provable, and probably no one ever will. This is the other side of the coin; like the crocodile paradox both are true! All of these sentences suffer from the same problem and the solutions that turn this paradox into deep theorems all suffer from the Blind Men and the Elephant Syndrome. They only look at one particular side/solution; the side that produces the deepest theorem, that we want to be true. It's not much different than the claim that an elephant is like a snake and ignore all other characteristics of the elephant. Whoever would have dreamed that so many deep truths lie underneath the problem of having two opposing views to be true at the same time? Or is it the case, that we only produce whatever truths we want via our intuition? Is this not empiricism but in other words and other means?

Appendix B: The Conditional in Truth or Truth in the Conditional : Limits of Truth Bias

This particular interpretation of truth as something that's not absolutely false, and the interpretation of the truth assignments in implication as "true unless shown by a counter example to be false" also gives some meaning to the so-called *vacuously true* statements of logic. For example, we find, in an excellent introductory book on logic [Barwise & Etchemendy, 1993]

One thing that often puzzles students has to do with the truth value of sentences of form

$$\forall x (P(x) \Rightarrow Q(x))$$

in worlds where there are no objects satisfying $P(x)$. If you think about it... you will see that in such a world the sentence is true simply because there are no objects that satisfy the antecedent. This is called a *vacuously true* generalization. Consider, for example, the sentence

$$\forall y (\text{Tet}(y) \Rightarrow \text{Small}(y))$$

which asserts that every tetrahedron is small. But imagine that it has been asserted about a world in which there are no tetrahedra. In such a world the sentence is true simply because there are no tetrahedra at all, small, medium, or large. Consequently, it is impossible to find a counterexample, a tetrahedron which is not small. What strikes students as especially odd are examples like

$$\forall y (\text{Tet}(y) \Rightarrow \text{Cube}(y))$$

On the face of it, such a sentence looks contradictory. But we see that if it is asserted about a world in which there are no tetrahedra, then it is in fact true. But that is the only way in which it can be true: if there are no tetrahedra. In other words, the only way this sentence can be true is if it is vacuously true. Let's call generalizations with this property "inherently vacuous." Thus a sentence of the form $\forall x (P(x) \Rightarrow Q(x))$ is inherently vacuous if any world in which it is true is also a world in which $\forall x \neg P(x)$ is true. [*underlining mine*]

It's not surprising that students get confused by the definition of implication. It would be easier to make its definition clearer than to fix it since apparently it is the best that can be done in bivalent logic. However, this bias already in implication permeates it without the realization in some quarters that mathematical truths, theorems and proofs are then subject to the same weakness already in the conditional; that is, they are true only until shown by a counterexample to be false. This adds additional firepower to the view of falsificationism due to Popper. Unfortunately some of the deep theorems of logic and set theory are not even falsifiable let alone verifiable or confirmable. How does one falsify the statement that there exist unprovable truths? The same flip-flopping that occurs in the various guises of the liar paradox and conaristry takes over. One cannot say: "You prove X to be false, if you cannot prove it to be false then it must be true."

This is the Achilles heels of formalism. i.e. there are unknowable truths because they exist independently of our existence vs statements are true only in given systems. Truth or true statements are then either a part of our imaginary universe and exist independently of our existence or are only true if we can know/prove them to be true. This is like Conway's production of numbers by biasing the truth in a strange way. Since the empty set has no members no member of the empty set can be greater than any member of any non-empty set. Therefore if we were attempting a partial order of a given set of sets of integers [including the empty set of course] the empty set would seem to head the list. But it is also true that since the empty set has no members, no member of the empty set can be less than any member of any nonempty set. If we go even further, we can then even order two, three or any number of empty sets! One could hardly say that a

statement such as $\forall x(U(x) \Rightarrow B(x))$, say, "All unicorns are blue." is true because we can't check the entire universe to be absolutely true that there are no unicorns. Since formalism says that truths are relative to a given formal system, employing the same logic as before we have that "All unprovable truths of any formal system F are true" is also true if there are no unprovable truths in any system. And if that is true, then "there exists an unprovable statement" is true; nothing but an empty truth requiring nothing more than playing with the loopholes of logic provided by the definition of implication, and the wonderchild of set theory, the empty set! There's no need for either Gödel coding or modal logic, since we have an almost perfect case of diagonalization, indeed double diagonalization. For example, the conditional

1) IF A, THEN B

really implicitly says

2) IF A THEN
B
ELSE
B OR NOT(B)

Since we can write this as $(A \Rightarrow B)(\neg A \Rightarrow [B + \neg B])$ and since the second factor is always true it's equivalent to $A \Rightarrow B$. It's difficult sometimes to know if this equivalence is a derived concept or really a part of the definition. One way to treat the above is as a shorthand for a longer formula or to treat it as something analogous to the Einstein suppressed summation notation for tensors. In the sciences including mathematics, in the beginning stages, it's not clear whether the definitions were concocted to be able to prove the beginning trivial theorems or if these trivial theorems are really a part of the definition. In the case above, we can very easily define other types of 'conditionals', for example;

3) IF A THEN
B
ELSE
NOT(B)

which is, of course the XNOR i.e. $(A \Rightarrow B)(\neg A \Rightarrow \neg B)$. Similarly it's not too clear if the contrapositive is not really a part of the definition of the conditional since it's quite obvious that $(A \Rightarrow B)(\neg B \Rightarrow \neg A)$ computes to $A \Rightarrow B$ and so does $(A \Rightarrow B) + (\neg B \Rightarrow \neg A)$. If instead of constantly trying to pour out these symbols into words and remembering axioms we simply treated these statements as equalities or expressions as in any other algebra and considered the underlying algebra as more fundamental then we could derive results much like the fields of mathematics which have proven themselves so powerful and so useful over the centuries. It seems that the easiest thing to do would be to present the implication as either one of (or both of)

4) $(A \Rightarrow B)(\neg A \Rightarrow [\neg B + B])(\neg B \Rightarrow \neg A)$
5) $(A \Rightarrow B) + (\neg B \Rightarrow \neg A)$

In the above equations and the discussion that follows, algebraic logic is used to determine the truth of the propositions (which can be considered to be sentences with suppressed quantifiers the same way one can

use the Einstein suppressed summation for tensor notation). We have another explanation of the role of the empty set in logic from different source.

The logic originally developed by Aristotle for categorical statements recognized as valid a number of immediate inferences that are not recognized valid in modern logic. This discrepancy can be traced to the presupposition in Aristotelian logic that all subject and predicate terms designate nonempty sets; modern logic assumes no such thing.... The presupposition of nonemptiness limited the applicability of Aristotelian logic. Moreover, once this presupposition was abandoned, logic was greatly simplified. Thus Aristotelian logic was for the most part abandoned, and the extra inferences it recognized ... were banished from subsequent logical systems.

The acceptance of empty terms by modern logic does however, create a problem: What is the truth value of A-form statements with empty subject terms? For example, is the statement 'All submarines over a mile long are pink' true or false? In a sense it doesn't matter, since there aren't any such submarines; but the demands of generality and completeness require a decision. Modern logic achieves an important gain in simplicity and systematicity by the stipulation that all A-form statements with empty subject terms are true. Thus the statement 'All submarines over a mile long are pink' is true - as, surprisingly, is the A-form statement 'All submarines over a mile long are not pink', which appears to contradict it.

But the appearance of contradiction is illusory. 'Not' in the second statement expresses complementation, not negation. Where S is empty, 'All S are P' says merely that the empty set is a subset of the set P. 'All S are not P' says merely that the empty set is a subset of the complement of P. This is not a contradiction, since by convention of modern mathematics the empty set is a subset of any set. (The logical convention that all-A form propositions with empty subject terms are true is, in fact, all of a piece with this mathematical convention.) [Nolt & Rohatyn, p. 108, 1988]

The set of all sets which are not members of themselves doesn't exist since it would be a contradiction. But in set theoretic terms, if we are to follow the above prescription or description, this set is the empty set! Then by the vacuous truth producing machinery above; "The set of all sets which are not members of themselves is X" is true as is the statement "The set of all sets which are not members of themselves is not-X". Then since diagonalization produces so many deep truths, we can diagonalize on the diagonalization i.e. double-diagonalize; let X be "is a member of itself." Then we have even a deeper truth; "The set of all sets which are not members of themselves is a member of itself and is not a member of itself" which is the Liar Paradox in disguise. Whether it's logic, set theory or the whole idea of diagonalization, there is something seriously wrong with a field in which there are so many ad hoc fixes. Is "not inside" equivalent to "not-not-outside"? Surely not, since the inside and outside refer to inside and outside of something, and that something is neither inside nor outside of itself. It's Neither and/or Both! The problem of the empty set shows up again. But this is equivalent to "no members of B are non-members of A". And since the empty set has no members, then the empty set cannot have any members that are non-members of A. But this is yet another variation of the Liar Paradox! It's true when observed one way and false if it's observed from a different perspective. Yet another case of the singularity in logic which refuses to go away. By the same logical equivalence, then since the empty set has no members, none of its members are members of any set! We can look at this as a saddle-point, or a singularity, or an oscillation, or lack of value. So many interpretations are possible.

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