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Dynamic Weighted Averages: Nonlinearity and Fuzzy Logic in Prediction

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Abstract

Since the field was kicked off by the models due to Merton[1973] and Black and Scholes[1973], the mathematical models in options, futures, and derivatives straddle the range from stochastic differential equations using Ito [Gard,1988] or Stratonovich [1967] calculi, to Markovian processes and martingales, to solutions of partial differential equations of Fokker-Planck-Kologorov Kwok[1998], [Rebenato,1996], [Chriss,1997], [Briys,1998]. Almost forgotten in the mathematical sophistication of these models is the simple fact that these models can or do contain deterministic components since they are based on a first order differential equation originally solved by Langevin. Knowing the system evolution equations means that Kalman filtering can be used on such linear differential equations [Wells,1996]. Simulation methods and other time series methods are also applicable. Fuzzy logic, which is the topic of this short paper, in many ways is more intuitively comprehensible than probability theory and more amenable to intervention analysis and hybrid technical-fundamental techniques, as shown below.

Introduction

Dynamic processes are classically defined in terms of differential equations (DEs). Stochastic differential equations (SDE) are generally defined as differential equations involving random elements. Since stochastic processes describing natural phenomena are understood as families of ordinary functions. SDEs can be visualized as representing families of deterministic sample differential equations. The ultimate goal in the solution of SDEs is the complete statistical description of the output x(t) from the statistical knowledge of the coefficients, the initial conditions and the input. Often, however, "solving" an SDE means the determination of a limited amount of information about the solution process such as expectation, correlation function or spectral density. From a mathematical as well as a physical point of view, the characterization of SDEs is strongly dependent upon the manner in which the randomness enters into the equations. It is thus convenient to distinguish three basic types of SDEs [Siski,1967].

a) Random initical or boundary conditions

b) Random forcing functions

c) Random coefficients

The first two cases are fundamentally simpler than the case of stochastic coefficients because of the deterministic relationship of the statistical properties of the solution to the statistical properties of the elements of randomness.

Random Initial Conditions

The passage from deterministic DEs to SDEs is easiest when random elements enter only through initial conditions. If x(t) is a dynamic process satisfying the DE

1) $$\frac{dx}{dt} = f(x(t), y(t), t)$$

satisfying the stochastic initial conditions, $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, the solution is $x(t) = g(x_0, \dot{x}_0, t)$. Since, once started, the random variable x(t) develops according to the deterministic law described by the function g, the joint distribution function x(t) and its derivative $\dot{x}(t)$ can be found from the joint density function of x_0, and $\dot{x}_0$ by standard methods of change of variables [Papaulis,1984]. The density function of $x(t)$ is then obtained as a marginal distribution.

Random Forcing Function

To treat the case of a random forcing function, we consider again the sample functions or sample solution properties of the stochastic process described by (1) where y(t) is a random process. A special case of eq. (1) is

2) $$\frac{dx}{dt} = f(x(t), t) + G(x(t), t)wt)$$

where w(t) is a Gaussian white noise process which has the formal representation $w(t) = \frac{d\beta}{dt}$ where b(t) is a Wiener Process. A further specialization of eq.(2) is given by

3) $$\frac{dx}{dt} = f(t)x(t) + w(t)$$

This is the starting point of many investigations of Brownian motion. The equation above was first treated by Langevin in his description of Brownian motion where

4a) $$\langle w(t)\rangle = 0$$

4b) $$\langle w(t)w(\tau)\rangle = 2D\delta(t-\tau)$$

where d(t) is the Dirac delta function and the angular brackets <.> indicate averaging. Albert Einstein [1956] obtained the probability density for such a process and thus obtained the first diffusion type equation for the probability density of such a dynamic process. Fokker and Planck later generalized these results. Still later Kolmogorov gave a more rigorous mathematical exposition of the process and also derived the backward Kolmogorov equation. Eq. (3) in the Ito formulation is given by

5) $$dx(t) = f(t)x(t)dt + d\beta(t)$$

which in the integral representation is

6) $$x(t) - x(t_0) = \int_{t_0}^{t} f(s)x(s)ds + \beta(t)$$

It has an explicit mean-square solution

7) $$x(t) = G(t, t_0)x_0 + \int_{t_0}^{t} G(t, s)w(s)ds$$

where G(t,s) is the Green's function of the differential equation (3) without the white noise. If we consider only the particular solution

8) $$x(t) = \int_{t_0}^{t} G(t, s)w(s)ds = \int_{t_0}^{t} G(t, s)d\beta$$

we can use this equation to determine the moments of the solution, the first two being the most important. It is easy to show for a vector case, that

9a) $$\langle x(t)\rangle = \int_{t_0}^{t} G(t, s)\langle w(s)\rangle ds$$

9b) $$\langle x(t)x^T(s)\rangle = \int_{t_0}^{s}\int_{t_0}^{t} G(t, u)R_w(u, v)G^T(s, v)dudv$$

The second moment, $R_x(t, s) = \langle x(t)x^T(s)\rangle$, as defined in (9b) is called the *autocorrelation*. A special case occurs when the input (forcing) is stationary and the system is time-invariant. Then eq.(8) can be written as

10) $$x(t) = \int_0^{\infty} G(\tau)w(t-\tau)d\tau$$

The autocorrelation function matrix of x(t) is

11) $$R_x(t-\tau) = \int_{t_0}^{s}\int_{t_0}^{t} G(u)R_w(t-\tau-u+v) + G^T(v)dudv$$

Random Coefficients

The solution of DEs with random coefficients, can be best expressed via the use of operator formalism, as in

12) $$L(z)x(z) = f(z)$$

where L(z) is a linear differential operator. In order to solve this, we would like an inverse operator $L^{-1}(z)$ satisfying

13) $$L^{-1}(z)L(z) = L(z)L^{-1}(z) = I$$

so that

14) $$x(z) = L^{-1}(z)f(z)$$

Since $L(z)$ is a differential operator, intuitively it is clear that $L^{-1}(z)$ is an integral operator so that

15) $$x(z) = L^{-1}(z)f(z) = \int_{t_0}^{t} G(t, s)f(s)ds$$

The kernel of this operator is the Green's function, and is the same function as in eqs.(7) through (11). Intuitively the function can be computed as follows:

16) $$x(z) = Ix(z) = L(z)L^{-1}(z)x(z) = L(z)\int_{t_0}^{t} G(z, s)x(s)ds$$

Interchanging the order of differentiation and integration

17) $$x(z) = \int_{t_0}^{t} L(z)G(z, s)x(s)ds$$

Therefore from the sifting property of the Dirac delta function it must be the case that

18) $$L(z)G(z, s) = \delta(z-s)$$

This is the defining equation for the Green's function. The theory of random differential equations with random coefficients is connected with the concept of a random operator (random Green's function) which maps random initial conditions and random forcing onto solutions which are the random trajectories which are the solutions of the DE.

Mathematical Modeling in Finance

When the equations underlying the process are known, at least to some degree of confidence, that is to say, that a mathematical model exists, such as in physics or engineering, for example, [Hubey,1983], economics [Hubey,1991] or even population biology [Turelli,1977], [Berg,1993], the approaches are to use the model in various ways, for example, by adding noise to the model (usually a set of differential equations). The resulting equations give rise to stochastic equations which can be approached using the Ito (or Stratonovich) stochastic calculus, as can be seen in Baxter & Rennie [1996]; the older traditional Fokker-Planck (or Kolmogorov) equations for the probability densities, Jazwinski[1970] or Gardiner[1985]; mean-square calculus [Soong, 1973]; or Kalman filtering [Wells, 1996]. If there are equations in finance in which we have great confidence, such as the Black-Scholes model, then similar approaches may be used and that is the reason for the great outpouring of stochastic models in finance (specifically for derivatives) during the last couple of decades.

If nothing is known or presumed known then statistical approaches which exist by the legion are employed, for example in Jenkins & Watts, [1968], or Medhi [1982]. The concepts of filtering, and prediction are often employed with regard to time series analysis. In time series analysis *hybrid methods* may be used. In hybrid methods one could be expected to use inputs from humans to fix up results of purely 'technical analysis'. Standard time-series techniques such as AR[I]MA filtering still require other interventions such as, detrending, prewhitening, outlier techniques, identification techniques, variance change techniques and parameter change detection procedures. If such procedures are used, then we cannot naturally get chaotic bursts in data as occurs in the stock/securities market. And if such bursts were cyclic they would be too predictable. The choices seem to revolve around smoothing out the large scale oscillations at the cost of missing out on real oscillations or leaving them in the model and obtain spurious bursts. In some cases we would like to be able to use IF-THEN operations built right-into the processes where these might come from those having financial intuition but not statistical knowledge. Fuzzy logic is a possible candidate for such models. This kind data-handling would be useful in financial prediction models in which the modeler wants to incorporate extra-financial information such as political interruptions or wants to incorporate impulsive (one-time large-scale) forces into the time-series model. This may be desirable even if the method being used, such as Kalman filtering already presumably accounts for such data.

Prediction and Statistics

Prediction in the case where the equations of motion can be described by linear differential equations as in eq.(3) can be done in a variety of ways. In the context of digital computation and stochastic processes it is called filtering and one of the most popular methods is the celebrated Kalman-Bucy filter [Jazwinski,1970], [Wells,1996]. In estimation problems of this type one often sees the computation of the estimate of the covariance function [Jenkins & Watts, 1968] but it is possible to get more information about the process by computing a more general version of the covariance. If x(t) and y(t) are two random processes, the cross-correlation functions are

19) $$R_{xy}(k) = \sum_{i=0}^{n-k} x_i y_{i+k} \text{ and } R_{yx}(j) = \sum_{i=0}^{n-k} y_i x_{i+k}$$

where $0 \le k \le n-1$. These are straightforward extensions of eq. (9) for finite and discrete data. To obtain normalized versions, i.e. *correlation density*, we simply compute

20) $$N_{xy}(k) = \frac{R_{xy}(k)}{n-k}$$

$R_{xy}(0)$ is the covariance function if instead of x(t) and y(t) we use $x(t) - \mu_x$ and $y(t) - \mu_y$. For x=y, then we obtain the autocorrelation of the procesess, $R_{xx}(k)$, and from this we can obtain the variance, $R_{xx}(0)$. If we want the process to possess the same characteristics in the future as it does in the past, then we have to relate the n-point autocorrelation function, $R^{(n)}_{xx}(k)$, to the (n+1)-point autocorrelation function, $R^{(n+1)}_{xx}(k)$, i.e. we need relationships of form

21) $$x_{n+1} = \frac{h(R(k))}{x_{n-k+1}}$$

Since we now have n-1 such equations, we have to somehow use all of them to obtain the best estimate for x_{n+1}. In order to extend the process into the future, if the $\{x_n\}$ does not depend on any way on any other variable, then we can compute x_{n+1} such that the autocorrelation of n+1 points is in some way the 'same' as the autocorrelation of the first n points. We might express this idea mathematically via

22.a) $$R^{n+1}(k) = R^n(k) + N^n(k) \quad \text{or}$$

22.b) $$N^{n+1}(k) = N^{n}(k)$$

Both of these are equivalent and then provide a set of estimates of form

23) $$x_{n+1} = N(k)x_{n-k+1}$$

Therefore we can create a single estimate from this set simply by averaging them

24) $$\hat{x}_{n+1} = \frac{1}{n}\sum_{k=0}^{n-1} N(k)x_{n-k+1}$$

It should be noted that the prediction is from each x_i forward using the autocorrelation for that lag. A simulation for the series in Figure I (below) is shown in Figure II.

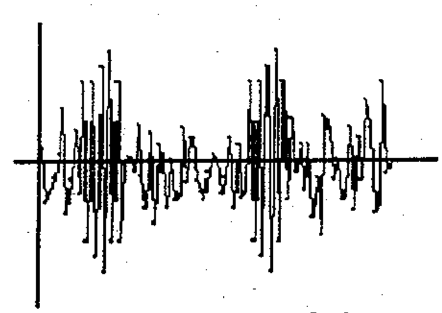

Figure I: A Simulated Time Series

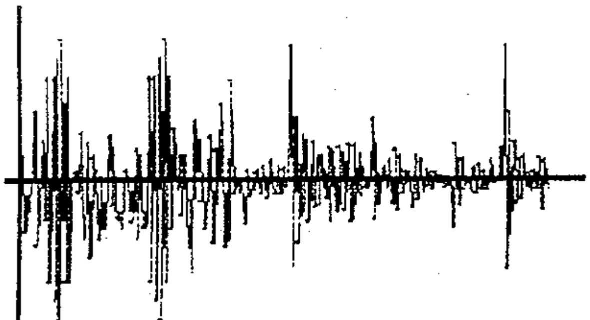

Figure II: Prediction Using the Autocorrelation

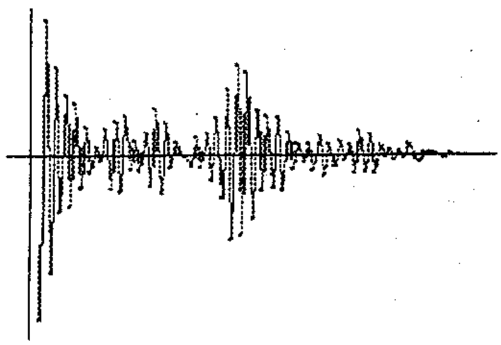

Figure III: Autocorrelation of the Simulated Series

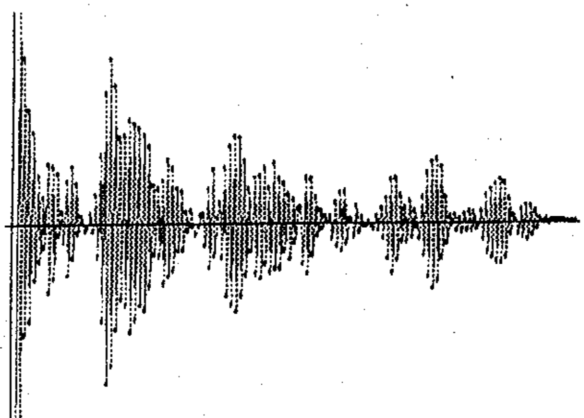

Figure IV: Autocorrelation of the Original + Prediction

It can be seen that the predicted values do not simply get averaged out but the chaotic bursts do carry forward. It can be seen in the autocorrelations that the characteristics of the original time series is exhibited by the predicted series.

Naturally, we would not really try to predict ahead twice as many periods as exists in our data, but we would rather recursively improve the estimates as the new data from the real world comes in as is done in Kalman filtering, and that can be easily accomplished also. There is only a single data point for $N_{xx}^{(n)}(n-1)$, thus the variance (hence the uncertainty) is high; there are n-1 data points for $N_{xx}^{(n)}(0)$, thus the variance (uncertainty) is low. This is one reason among many why we should count the earlier correlations less heavily i.e. this is simply a restatement of the Markovian assumption. However, if there are any cyclic correlations we would miss them if we used exponentially decreasing weighting. We might attempt somehow use the autocorrelation itself as the weighting function in determining the next data point. The advantage of using the autocorrelation itself is that it will capture the characteristics of its past (especially if there are a sufficient number of data points). The disadvantage is that it would capture its past only too well, and might produce cyclic correlation based only one some data points. In real life the behavior of one system cannot really be easily isolated from the rest so that we should consider a set of such related processes meaning that should use the matrix of crosscorrelations given by;

25) $$R(\tau) = \begin{bmatrix} R_{x_1x_1}(\tau) & R_{x_1x_1}(\tau) & \ldots & R_{x_1x_n}(\tau) \\ R_{x_2x_1}(\tau) & R_{x_2x_2}(\tau) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ R_{x_nx_1}(\tau) & \ldots & \ldots & R_{x_nx_n}(\tau) \end{bmatrix}$$

Fuzzy Logic

In recent years fuzzy logic has been promoted as an alternative to probability theory in reasoning under uncertainty. A version of fuzzy-logical implication (many polynomial forms of which can be found in Hubey [1999]) in combination with dynamic averaging can be used as input into the process using guesses by experts to make up for the tendency of the various prediction and filtering mechanisms to recreate the past only too well. The fuzzy-logic version is superior to IF-THEN type of programming since instead of {0,1}, using [0,1] we can use the fuzzy values as a kind of a smoothing operator and as an extension of [exponential] weighting. Furthermore since many of the fuzzy-logics in Hubey [1999] are algebraic (i.e. polynomial) they can be incorporated into the averaging algorithm much more easily and since the fuzzy implication IF-THEN is used not as the crisp implication but rather as a command to execute if the antecedent is true, we can use the values of the variables themselves in the IF-THEN therefore creating a truly nonlinear algorithm, approximating the true state of the world.

For example, we might be creating many time-series from many raw time series. These processed series might be things such as risk-factor, threshold-factor, etc.

26.a) *IF* $x(t) > 120$ *THEN threshold:=1.*
26.b) *IF* $x(t) > \langle x(t)\rangle_n >$ *THEN risk_factor :=risk_factor+1*

where $<x(t)>_n$ is the running average for the past n periods. Eq. (26a) could easily be implemented using fuzzy-logics. Probably the simplest of such logics is the *bang-bang logic* [Hubey,1999] which can be obtained by noting that logical implication $P \Rightarrow Q$ is false whenever P > Q. Thus using the standard Heaviside Unit Step function

27) $$U(x-x_0) = \begin{cases} 1 & x \geq x_0 \\ 0 & x < x_0 \end{cases}$$

we can construct the implication $P \Rightarrow Q$ as $U(Q-P)$. Since the Heaviside function can also be written in terms of polynomials [Hubey,1999] it can be used directly in the calculations. Furthermore, instead of U(x) we can use any of the more smoothly varying norms of fuzzy logic. For example, suppose we want to create a running average but want to ignore outlying points which we define as points which are b times as large as the running average. The running average is obviously calculated iteratively via

28) $$\langle x_{n+1}\rangle = \frac{\langle x_n\rangle n}{n+1} + \frac{x_{n+1}}{n+1}$$

If we do not want the x_{n+1} term to enter into the new average but want to ignore it if $x_{n+1} > b<x_n>$ then we can write this as

29) $$\mu_{n+1} = U(\beta\mu_n - x_{n+1})B + U(x_{n+1} - \beta\mu_n)\mu_n$$

$$where \quad B = \frac{n\mu_n}{n+1} + \frac{x_{n+1}}{n+1}$$

As an iterative/recursive series this is a simple example of nonlinearity built into the equations and which have meaning as fuzzy implication. Since the diffusion processes in derivative models seem to have a tendency to possess leptokurtosis, and also have larger variances around the mean, we might take this into account in Monte Carlo simulation however it is known that random number generators produce correlated output Marsaglia [1968], Park & Miller [1988], MacLaren & Marsaglia [1965], Hubey & Gutierrez [1997], Peterson [1991].

One can obviously create more complex series such as finding the moving average of the last m days, or evaluating functions depending on last day's value, or more complex functions of other variables from the past. The use of such fuzzy values is an improvement over the computation of the autocorrelation or crosscorrelation functions which lose much information that is in the data or attempting to code IF-THEN statements into mathematico-statistical computations. The use of various types of fuzzy logics which do not change so rapidly as in the threshold type above (bang-bang function) can be used as more focused smoothing and averaging operators on time series. A particularly simple fuzzy logic that has smooth versions can be constructed from the two functions, Hubey[1999];

30) $$H_h(x,y) = \frac{(x+y)^h}{2} \quad M_m(x,y) = 2^{m-1}\left[\frac{(x-y)^2}{2(\{x-y\}^2)^{0.5}}\right]^m$$

Then from these functions we can create the standard fuzzy sets (logical connectives) as;

31.a) $$OR_{1,m}(x,y) = u_{1,m}(x,y) = H_1(x,y) + M_m(x,y)$$

31.b) $$AND_{1,m}(x,y) = i_{1,m}(x,y) = H_1(x,y) - M_m(x,y)$$

Both are idempotent and continuous. The now-standard Max(x,y), and Min(x,y) functions of Zadeh for union and intersection are $OR_{11}(x,y)$ and $AND_{11}(x,y)$ respectively. For $m \neq 1$ smooth functions can be obtained. Fuzzy implication can be constructed from these in many ways [Hubey,1999] and used in smoothing and prediction operators and in injecting exceptional IF-THEN rules into the computations. It would be possible to use the above norm in combination with genetic/evolutionary algorithms to compute the values of the parameters *m* and *h* that best predict a given time-series.

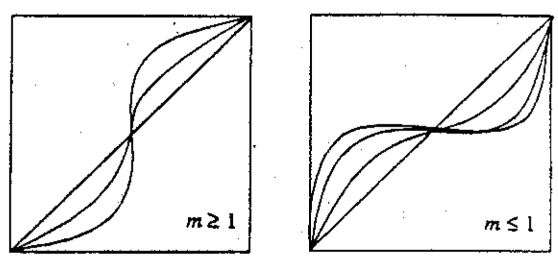

Figure V: Fuzzy Norms from Eq.30 [Hubey,1999]

Conclusion and Discussion

A large class of estimation problems concerns itself with finding an optimal estimate of some quantity (an unknown parameter, a random variable or an uncertain quantity) when a linear function of this quantity is corrupted by noise (usually taken to be additive noise). One of the first studies of this type of problem was done by Gauss in the early

1800's in which he introduced the least-squares estimates of unknown parameters. In the early 1940s Wiener and Kolmogorov attempted solutions of a class of problems dealing with estimation of random signals. The so-called Wiener-Hopf integral equation is an equation for the weighting function which when convolved with the noisy measurements produces an unbiased minimum variance estimate of the random signal. The equation can only be solved for certain special cases of the general problem and thus had limited practical application unless solved numerically. In the 1950s the idea of recursively computing least-squares estimates was [re]introduced.

The paper by Kalman[1960] introduced a different approach to the problem of Wiener and Kolmogorov by changing the Wiener-Hopf integral equation into a differential equation, and then showed a recursive procedure for estimation of the solution instead of looking for an analytical solution. Since many of the methods involving derivatives books such as Rebenato [1996], Briys [1998], or Kwok[1998] are solutions of the diffusive type (Fokker-Planck-Kolmogorov equations) and which are obtained from a linear equation of the Langevin type, then Kalman filtering can be used for these problems as well. However, the main problem as always in economics and finance is in not knowing the sources of uncertainty and the exact relationships posited by the simple linear differential equation from which the diffusion equations are obtained. Instead one can use the data that is suspected of being correlated to extend the data into the future or use this data and/or extension to make estimates of the coefficients of the process. Information that is relevant but does not show up in the equations can be put into the estimation procedure by using fuzzy logical reasoning which is much more intuitively comprehensible than is the theory of stochastic differential equations.

There is always the problem of recreating the past only too well. In these cases, nonlinear methods must be used. For example, only nonlinear equations can create the nonperiodic (chaotic) bursts observed in many financial data. It is already known that the stochastic methods in use create Gaussian distributions whereas real data has fatter tails and greater concentration around the mean so that the volatility is greater around the mean and in the neighborhood of the extremes, see for example Chriss [1997:115]. It is easy to show that the sum of two Gaussians, one with a very small variance and another with a large variance can produce the same kind of output as that observed in empirical evidence. This means most likely that there are two sources of noise, and that the system switches between these two regimes, and this can be most easily modeled via nonlinear equations.

It is interesting that while there are competing stochastic calculi, namely the Ito vs. Stratonovich calculi, [Fox,1976] for making rigorous the intuitive results of Langevin, Einstein, Fokker and Planck discussed above, the literature in finance reveals only Ito. Ito (inspired by Doob) and later Stratonovich extended the white-noise stochastic equations to apply to multiplicative type stochastic equations [Fox,1976], [Stratonovich, 1967], Van Kampen [1976], Hubey[1983]. The martingale property of the Ito equation as observed by Stratonovich is a mathematical nicety that is not necessary on other grounds. It is known that similar deterministic equations, viewed as approximations of stochastic models lead to different limiting diffusion equations, and that there is a controversy surrounding such models [Gard,1988:167]. The Stratonovich integral corresponds more to physical reality as reported in West[1979], and Bulsara [1979]. Furthermore, the Markovian property is also a mathematical nicety with no physical basis, and the extensions to the theory of stochastic differential equations made by van Kampen [1976], and others allow the solution of non-Markovian Gaussian problems which correspond more with reality [Fox,1976], van Kampen [1976]. Realistic models should include auto and cross-correlated noise instead of the simple delta-correlated forcing. Diffusive approximations for such processes have been obtained [Gard,1988:163]. The theory of multiplicative stochastic processes is based on the utilization of all higher order moments and has specifically time-ordered cumulant averages Kubo [1962,1963], Fox [1976,1977,1978], Van Kampen [1974]. Since these problems require the computation of higher order moments McKenna [1970], various truncation schemes Adomian [1971], Bellman [1972] and projection operator techniques LoDato[1973], Terwiel [1974] among others Adomian [1970] have been employed. Many of these methods can be employed in hybrid forms which can be especially suitable the financial field if connected together via intuitive fuzzy logic.

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