Fair Allocations in an Overlapping Generations Economy *

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Abstract

The paper examines the nature of fair (envy free, efficient) allocations in an overlapping generations economy without production, in which each generation lives for two periods. It shows that there exists a non-trivial conflict between equity and efficiency when all generations have identical preferences. This conflict is seen to be entirely determined by the historically determined young age consumption of the generation which is old in the first period when the social decisions are being made, relative to the young age consumption in the golden-rule. It then shows by an example that there could exist non-stationary preferences for which such a conflict does not arise regardless of the history of the economy.

Keywords: Overlapping Generations, Envy-Free, Efficiency, Fair Allocation.
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1 Introduction

Consider the following problem for a policy maker. At time $t = 1$, (present), there live agents of different ages, who can be broadly categorized as young and old. The old have already lived out their youth in the past and had some consumption (call it x_0). The policy maker needs to decide on ways of allocating a fixed amount of resource, each period,

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among the young and old from the present into the infinite future. What can be an egalitarian allocation for this economy? Can such an equitable allocation be achieved efficiently? What is the nature of constraints imposed on the feasible egalitarian policies by the fact that x_0 is already determined in the past? This paper is an attempt to frame the debate on the class of questions posed above. It is thus a contribution to the social decision theory literature.

In the theory of intertemporal social choice, two approaches have been used in studying efficient equitable allocations over time. The first approach seeks to define social preferences in the form of a social welfare quasi-order, a social welfare order or a social welfare function on infinite utility streams, which embodies in it the notions of equity and efficiency. Then, given an economy, these preferences can be used to choose among the utility streams over time that the economy is capable of generating. The choice set is the set of allocations which generate utility streams that are maximal in terms of the social preferences[1].

An alternative approach is to start with an economy and obtain a description of its feasible allocations. The choice set is a set of allocations obtained by imposing notions of equity and efficiency directly on the choice rules[2].

The notion of efficiency in both approaches involves some version of the Pareto principle. The notion of equity that is used varies widely, but the notions of equity used in the first approach[3] can be translated to notions of equity in the second approach.

The particular concept of equity that we wish to study in this paper is known as *envy-free*, which was introduced in the economics literature by Foley (1967)[4]. Unlike other notions of equity, it is quite essential to start with a description of the feasible allocations of an economy to even define this concept. This is because the notion of being envy-free involves each generation comparing its own *assignment* in an allocation to the *assignments* of all other generations in that allocation. Thus, our study follows the second approach described above. Also, unlike other notions of equity, there need be no comparability requirements placed on the utility functions of the various generations, since each generation evaluates its own assignment and the assignment of every other generation, in terms of its own utility function.

Shinotsuka et al. (2007) have studied alternative notions of envy-free allocations in an overlapping generations economy without production, in which each generation lives

[1]For some recent contributions following this approach, see Mitra (2005), Basu and Mitra (2007), Banerjee and Mitra (2010), Asheim et al. (2010b).

[2]For recent contributions following this approach, see Shinotsuka et al. (2007), Asheim et al. (2010a).

[3]Two concepts of equity that have been discussed extensively are *Anonymity* and the *Pigou-Dalton principle*.

[4]Other seminal references on this equity notion are Kolm (1972), Varian (1974), and Yaari (1981). Some of the recent surveys on envy free allocations are Thomson (2010).

for two periods. We retain their framework, but assume in addition that the population is stationary and that there is a social endowment of a unit of a perishable consumption good in each period to be divided between the young and the old living in that period. This simplification helps us to highlight the essential issues involved.

We examine in this framework the notion of envy-free allocations that appears to us to be the most persuasive; also, unlike their focus on stationary allocations, we study all allocations, stationary or not. We then analyze the possible conflict between equity and efficiency considerations in determining the social choice set[5].

The first part of the paper focuses on the case of *stationary* preferences. The possible conflict that can arise between efficiency and equity considerations can be seen with a variety of specifications of the (identical) utility function of the generations. However, to stress that this conflict is not pathological, we deliberately specify the utility function to be "well-behaved" in every way, and such that the economy has a unique golden-rule in the interior of the consumption space.

Given the golden-rule, the conflict is then seen to be entirely determined by *history*. That is, it is determined entirely by the young-age consumption of generation 0 (call it x_0), who is alive and old in period 1, when social choices are made about that period and all future periods. At the time of choice, then, x_0 is given, having been determined and already carried out in the past.

The main result (Theorem 1) is that if this historically determined young-age consumption (x_0) is *high* (relative to the golden-rule young-age consumption $\bar{x}$), then there is no allocation which is efficient and envy-free. If it is *low* (relative to the golden-rule young-age consumption $\bar{x}$), then there always exists an efficient and envy-free allocation (called a *fair* allocation).

The conflict may be seen as follows. If $x_0 > \bar{x}$, it is possible to ensure golden-rule utility for all generations, and provide a bit more utility to generation 0. It is not possible to ensure any higher *constant* utility for all generations than golden-rule utility. Then, equity considerations force one to choose golden-rule utility for all generations; but this is clearly Pareto inefficient.

In the second part (Section 4), we consider the case of non-stationary preferences. Intuitively it is not difficult to come up with economies (i.e., diverse agents' preference and / or endowments, etc.) exhibiting conflict between equity and efficiency over larger subsets of possible histories. However, what is not obvious is whether, for every specification of (non-stationary) preferences, there will be some history for which a conflict between equity and efficiency will always exist. We construct a specification of preferences for which no such conflict between equity and efficiency arises, regardless of the history of

[5]The notion of Pareto efficiency that we use is also somewhat different from Shinotsuka et al. (2007). This is explained in Section 2.

the economy. It should be noted that since the social endowment for the economy remains the same as before, the result is obtained by introducing diversity in the preferences of the agents. However, in order that such an example be of interest, we need to retain the well-behaved nature (as in Theorem 1) of each agent's preferences.

The analysis of the case of non-stationary preferences is essentially different from the case of stationary preferences (examined in Section 3) in that there is no natural benchmark like the golden-rule allocation to compare with any given allocation. In order to still avail of a pseudo benchmark, the kind of preferences used in the example that we construct involve stationary preferences on a subset of the domain, and non-stationary preferences on the remainder, with the domain of non-stationarity becoming very "small" as we consider generations far into the future.

For future research, we intend to examine the case of agents who live for more than two periods. The economy with three period lived agents would be an appropriate model as the economy with two period lived agents could be restrictive.[6]. It is well-known that there exist stationary economies, where agents live for more than two periods, for which there are no efficient stationary allocations satisfying individual rationality. Shitovitz (1988) constructs an economy with three periods lived agents having standard additively separable Cobb Douglas preferences and shows that economy fails to have any stationary efficient allocations[7]. In our model with two-period lived agents, essential use is made of the existence of stationary efficient allocations in establishing that there is no conflict of equity with efficiency when the historically determined young age consumption is less than the golden-rule young age consumption. The role of the life span assumption is therefore crucial and it would be interesting to investigate the nature of conflict between equity and efficiency in a three periods overlapping generations economy.

2 An Overlapping Generations Model

2.1 Allocations

We consider a standard overlapping generations model in which for each $t \in \mathbb{M}$, a single agent is born (called generation t) who lives for two periods (t and $t+1$)[8]. Therefore, in

[6]See Samuelson (1958) for three periods lived agents, Gale (1973) for the general finite n-period lived agents, Cass and Yaari (1966).

[7]The non-existence result persists even when stationary allocations are required to satisfy a weaker notion of efficiency, namely "Weak Pareto Optimality" as defined by Balasko and Shell (1980); see Okuno and Zilcha (1983).

[8]The set of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{M} \equiv \mathbb{N} \cup \{0\}$.

each period $t \in \mathbb{N}$, there is an old agent who overlaps with an agent in his youth[9].

For each generation $t \in \mathbb{M}$, we denote by x_t the generation's consumption when young, and by y_{t+1} the generation's consumption when old.

In each period $t \in \mathbb{N}$, one unit of an intrinsically desirable but completely perishable good is made available exogenously. This has to be distributed between the young and old alive in period t. We consider $\bar{x}_0$ (the consumption of generation 0 when young) to be historically given, with $\bar{x}_0 \in [0,1]$. Thus, we view social decisions being made at date 1 regarding assignments of the single good to the young and old in period 1, and in all subsequent periods. Decision making at period 1 clearly cannot affect $\bar{x}_0$, which has been determined in the past.

An *allocation* is a sequence $\langle x_t, y_{t+1} \rangle = \langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ satisfying,

$$\left.\begin{array}{lll} (x_t, y_{t+1}) & \geqslant 0 & \text{for all } t \in \mathbb{M}, \\ x_t + y_t & \leqslant 1 & \text{for all } t \in \mathbb{N}, \\ x_0 & = \bar{x}_0. & \end{array}\right\} \tag{1}$$

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *stationary* if,

$$(x_t, y_{t+1}) = (x_{t+1}, y_{t+2}) \quad \text{for all } t \in \mathbb{M}. \tag{2}$$

2.2 Equity and Efficiency

Even though part of the consumption bundle of generation 0 is historically given, the well-being of generation 0 can clearly be affected by decisions made at time period 1 (that is, by the assignment of its old-age consumption, y_1). Thus, the notions of both equity and efficiency have to be formulated so that the well-being of generation 0 (and of course the well-being of all generations from $t = 1$ onwards) are taken into account.

Assume that the preferences of generation t (where $t \in \mathbb{M}$) on consumption bundles in $X \equiv \mathbb{R}_+^2$ are expressed by a utility function u_t from X to $\mathbb{R}$.

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *envy-free* if for each $t \in \mathbb{M}$,

$$u_t(x_t, y_{t+1}) \geqslant u_t(x_s, y_{s+1}) \quad \text{for all } s \in \mathbb{M}. \tag{3}$$

An allocation $\langle x'_t, y'_{t+1} \rangle_{t=0}^{\infty}$ *dominates* an allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ if,

$$u_t(x'_t, y'_{t+1}) \geqslant u_t(x_t, y_{t+1}) \quad \text{for all } t \in \mathbb{M}, \tag{4}$$

with strict inequality in (4) for some $t \in \mathbb{M}$. An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *efficient* if there is no allocation which dominates it.

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *fair* if it is both envy-free and efficient.

[9] This is a standard overlapping generations model. Balasko and Shell (1980) is the standard reference.

3 The Case of Stationary Preferences

In this section, we specialize to the case in which the preferences of the generations are stationary; that is, there is $u : X \to \mathbb{R}$ such that,

$$u_t = u \text{ for all } t \in \mathbb{M}. \tag{5}$$

We will proceed to make assumptions on u which ensure that the preferences are "well-behaved" in every way, so that the conflict between equity and efficiency considerations, which we will demonstrate in Theorem 1 below, is not seen as arising from unusual specification of preferences.

The utility function $u : X \to \mathbb{R}$ will be assumed to satisfy,

A1. (Continuity) u is continuous on X,

A2. (Quasi-Concavity) u is quasi-concave on X, and

A3. (Monotonicity) u is monotone on X and strongly monotone on $\mathbb{R}^2_{++}$.

3.1 A Golden Rule

Formally, a golden-rule is a pair $(\bar{x}, \bar{y})$ which solves the following problem

$$\left.\begin{array}{ll} \text{max} & u(x,y) \\ \text{subject to} & x+y \leqslant 1 \\ \text{and} & (x,y) \geqslant 0. \end{array}\right\} \tag{GR}$$

It can be interpreted in the following way. Given any historically given $\bar{x}_0 \in [0,1]$, consider the stationary allocation $\langle x, y \rangle$ which does not waste any of the endowment. This is given by $\langle \bar{x}_0, 1-\bar{x}_0 \rangle$ with stationary utility of $u(\bar{x}_0, 1-\bar{x}_0)$. Now, consider all economies possible with different historically given levels of $\bar{x}_0 \in [0,1]$. Pick the economy which has the largest utility level $u(\bar{x}_0, 1-\bar{x}_0)$ among all such economies. This corresponds to the golden-rule.

We will assume that u is such that,

A4. There is a unique solution $(\bar{x}, \bar{y})$ to problem (GR), and $(\bar{x}, \bar{y}) >> 0$.

It follows, of course, from A3 and A4 that

$$\bar{x} + \bar{y} = 1. \tag{6}$$

Let us define the sets

$$\begin{aligned} A &= \{(x,y) \in [0,1] \times [0,1] : u(x,y) \geqslant u(\bar{x}, \bar{y})\}, \\ B &= \{(x,y) \in [0,1] \times [0,1] : u(x,y) > u(\bar{x}, \bar{y})\}. \end{aligned} \tag{7}$$

It follows from the definition of a golden-rule that

$$x + y > 1 \ \text{ for all } (x,y) \in B, \tag{8}$$

and it follows from the definition of a golden-rule and A3 that

$$x + y \geqslant 1 \ \text{ for all } (x,y) \in A. \tag{9}$$

A useful property about the golden-rule can now be noted.

Lemma 1. *Given any* $\varepsilon > 0$, *there is* $\delta > 0$ *such that whenever* $(x,y) \in A(\varepsilon)$*, where*

$$A(\varepsilon) = \{(x,y) \in [0,1] \times [0,1] : u(x,y) \geqslant u(\bar{x},\bar{y}) + \varepsilon\},$$

we must have

$$x + y \geqslant 1 + \delta \tag{10}$$

Proof. The set $A(\varepsilon)$ is non-empty (since $(x,y) \in A(\varepsilon)$), bounded and closed (by A1). The function, $h : A(\varepsilon) \to \mathbb{R}$, defined by $h(x,y) = x + y - 1$ for all $(x,y) \in A(\varepsilon)$, is continuous and positive on $A(\varepsilon)$ (by (8)). It attains a minimum on $A(\varepsilon)$. Denoting this minimum value by δ, (10) is satisfied and further we have $\delta > 0$. □

Now, let $(\hat{x},\hat{y}) \in \mathbb{R}^2_{++}$ satisfy $\hat{x} + \hat{y} = 1$, with $\hat{x} < \bar{x}$. Then, by (6), $\hat{y} > \bar{y}$. Let us define the sets

$$\begin{aligned} \hat{A} &= \{(x,y) \in [0,1] \times [0,1] : u(x,y) \geqslant u(\hat{x},\hat{y})\}, \\ \hat{B} &= \{(x,y) \in [0,1] \times [0,1] : u(x,y) > u(\hat{x},\hat{y})\}. \end{aligned} \tag{11}$$

Lemma 2. *There exist* $p > q > 0$ *such that*

$$px + qy \geqslant p\hat{x} + q\hat{y} \ \textit{ for all } (x,y) \in \hat{A}, \tag{12}$$

and

$$px + qy > p\hat{x} + q\hat{y} \ \textit{ for all } (x,y) \in \hat{B}. \tag{13}$$

Proof. Define the set

$$\bar{A} = \{(x,y) \in \mathbb{R}^2 : (x,y) << (\hat{x},\hat{y})\}$$

Clearly, $\hat{A}$ and $\bar{A}$ are non-empty, convex sets in $\mathbb{R}^2$ (by A2), and $\hat{A}$ is disjoint from $\bar{A}$ (by A3). Thus, by a standard separation theorem we can obtain $(p,q) > 0$, such that (12) is satisfied. Since $(\hat{x},\hat{y}) >> 0$, we can use A1 (continuity) and A3 (strong monotonicity) to ensure that $(p,q) >> 0$. Further, one can use A1 to ensure that (13) is satisfied as well.

It remains to verify that $p > q$. By A4, we must have $u(\bar{x},\bar{y}) > u(\hat{x},\hat{y})$, so by (13),

$$p\bar{x} + q\bar{y} > p\hat{x} + q\hat{y}$$

Using (6) and the fact that $\hat{x} + \hat{y} = 1$, we obtain

$$p(\hat{y} - \bar{y}) > q(\hat{y} - \bar{y})$$

and since $\hat{y} > \bar{y}$, we must have $p > q$. □

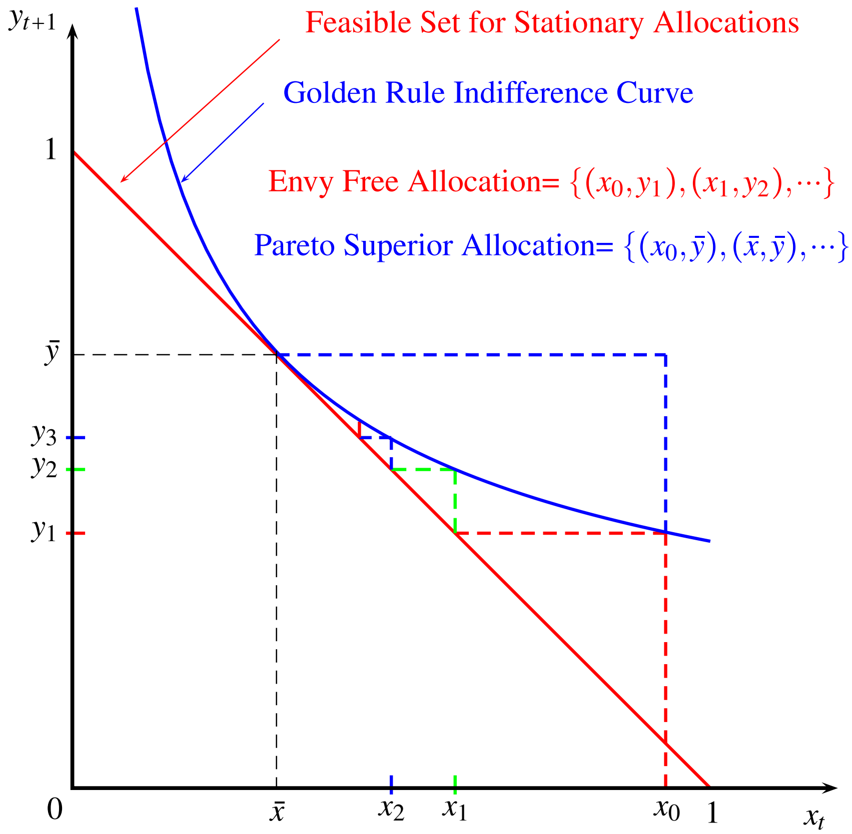

Figure 1: Necessary Condition for Fair Allocation

3.2 Main Result

We are now in a position to state and prove the main result on stationary preferences.

Theorem 1. *(i) Suppose $\bar{x}_0 > \bar{x}$. Then, there is no fair allocation.*
(ii) Suppose $0 < \bar{x}_0 \leqslant \bar{x}$. Then, $\langle x_t, y_{t+1} \rangle$ defined by,

$$(x_t, y_{t+1}) = (\bar{x}_0, 1 - \bar{x}_0) \ \ \textit{for all } t \in \mathbb{M},$$

is a fair allocation.

Proof. (i) Suppose, on the contrary, there is a fair allocation, call it $\langle x_t, y_{t+1} \rangle$. Since it is envy-free, we must have $u(x_t, y_{t+1})$ constant for all $t \in \mathbb{M}$. Denote this constant utility level by v. We claim that $v \leqslant u(\bar{x},\bar{y})$. For if $v > u(\bar{x},\bar{y})$, then denoting $v - u(\bar{x},\bar{y})$ by ε, we have $\varepsilon > 0$, and by Lemma 1, there is $\delta > 0$, such that,

$$x_t + y_{t+1} \geqslant 1 + \delta = \bar{x} + \bar{y} + \delta \text{ for all } t \in \mathbb{M}. \tag{14}$$

Noting that $x_t + y_t \leqslant 1$ for all $t \in \mathbb{N}$, we then obtain,

$$(y_{t+1} - \bar{y}) \geqslant (y_t - \bar{y}) + \delta \text{ for all } t \in \mathbb{N}. \tag{15}$$

But, (15) implies that $(y_{t+1} - \bar{y}) \to \infty$ as $t \to \infty$, which contradicts the fact that $y_{t+1} \leqslant 1$ for all $t \in \mathbb{M}$. This establishes the claim.

Consider now the sequence $\langle x'_t, y'_{t+1} \rangle$ defined by

$$x'_0 = \bar{x}_0, (x'_t, y'_t) = (\bar{x}, \bar{y}) \text{ for all } t \in \mathbb{N}. \tag{16}$$

It is easy to check then that $\langle x'_t, y'_{t+1} \rangle$ is an allocation. Further,

$$u(x'_t, y'_{t+1}) = u(\bar{x},\bar{y}) = v \geqslant u(x_t, y_{t+1}) \text{ for all } t \in \mathbb{N}$$

and

$$u(x'_0, y'_1) = u(\bar{x}_0, \bar{y}) > u(\bar{x},\bar{y}) = v \geqslant u(x_0, y_1),$$

establishing that $\langle x_t, y_{t+1} \rangle$ is inefficient. This contradiction establishes the result.

(*ii*) Clearly, $\langle x_t, y_{t+1} \rangle$ defined by

$$(x_t, y_{t+1}) = (\bar{x}_0, 1 - \bar{x}_0) \text{ for all } t \in \mathbb{M}, \tag{17}$$

is an allocation. It is also envy-free. It remains to show that it is efficient. Suppose, on the contrary, there is an allocation $\langle x'_t, y'_{t+1} \rangle$ such that,

$$u_t(x'_t, y'_{t+1}) \geqslant u_t(x_t, y_{t+1}) \text{ for all } t \in \mathbb{M}, \tag{18}$$

with strict inequality in (18) for some $t \in \mathbb{M}$. Then, by (12),

$$px'_t + qy'_{t+1} \geqslant p\bar{x}_0 + q(1 - \bar{x}_0) \text{ for all } t \in \mathbb{M}. \tag{19}$$

Since $x'_0 = \bar{x}_0$, (19) implies that,

$$(y'_1 - y_1) \geqslant 0, \tag{20}$$

and

$$(y'_{t+1} - y_{t+1}) \geqslant (p/q)(y'_t - y_t) \text{ for all } t \in \mathbb{N}. \tag{21}$$

Since there is strict inequality in (18) for some $t \in \mathbb{M}$, we must have strict inequality in (20), or in (21) for some $t = T \in \mathbb{N}$. In either case, there is some $s \in \mathbb{N}$ for which $(y'_s - y_s) > 0$. Then using (21) for all $t \geqslant s$, and noting that $(p/q) > 1$, we have $(y'_{t+1} - y_{t+1}) \to \infty$ as $t \to \infty$. This contradicts the fact that $y'_{t+1} \leqslant 1$ for all $t \in \mathbb{M}$, and establishes the efficiency of $\langle x_t, y_{t+1} \rangle$. □

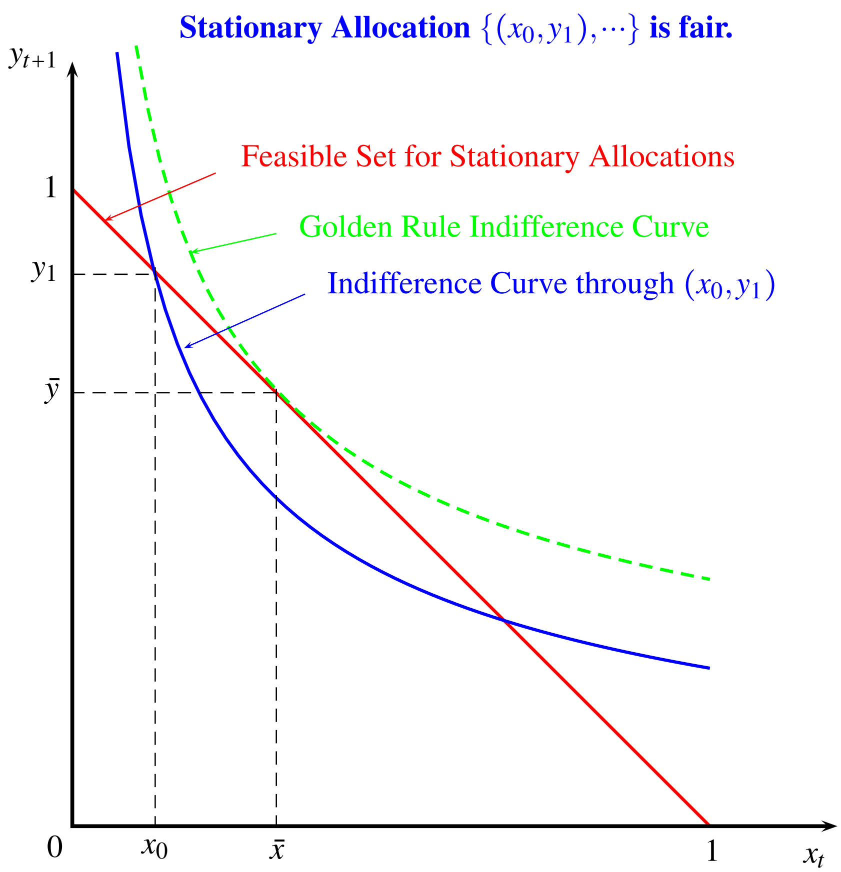

Figure 2: Sufficient Condition for Fair Allocation

4 Non-Stationary Preferences

In this section we show, by an example, that there exists a set of non-stationary preferences which admits a fair allocation for all possible histories.

We explain the logic of this example as follows. We choose the set of agents' preferences and a feasible set in the following manner. First we enlarge the feasible allocation set from $x+y \leq 1$ (in the stationary case) to $x+2y \leq 2$.[10] In order to specify the preference for agent t, we carry out following thought experiment. Let there be a stationary economy with all agents being similar to agent t and the resource constraint be $x+2y \leq 2$. In this economy, there exist golden rule allocations (not necessarily unique) for the preferences satisfying A1-A3[11]. The choice of assignment for agent t, $(\bar{x}_t, \bar{y}_{t+1})$, would then be such that $\bar{x}_t$ is the minimum young age consumption among all the young age consumptions in the set of all golden rule allocations for agent t.

Thus we have a well-defined unique assignment for each agent. The set of preferences is such that $(\bar{x}_0, \bar{y}_1) = (1, \bar{y}_1)$ where $\bar{y}_1 \in (0,1)$, $\bar{x}_t = 1 - \bar{y}_t$ and $\bar{y}_{t+1} \in (\bar{y}_t, 1)$ for all t. The allocation further satisfies the condition that the sequence of young age consumptions, $\langle \bar{x}_t \rangle$, is monotone decreasing to 0 and the sequence of old age consumptions, $\langle \bar{y}_t \rangle$, is monotone increasing to 1. This implies that the allocation is feasible for the history $\bar{x}_0 = 1$. This allocation plays the role of a pseudo benchmark similar to the role played by the golden rule allocation in the stationary economy. In the example, we show that this pseudo benchmark is an envy free and efficient allocation. We further restrict the preferences such that they become stationary over larger subset of domain $x_t \in [0,1]$ as we go out in the future[12]. For $x_0 < 1$, we show that there exists a fair allocation which deviates from the pseudo benchmark allocation for only finitely many initial agents.

Example 1. Let the utility function for generation $t \in \mathbb{M}$ be

$$u_t(x,y) = \begin{cases} \left(\frac{2^{t+2}}{1+2^{t+1}}\right) \cdot (x+y) & \text{for } y \geqslant (2^t - 0.5)x, \\ x+2y & \text{otherwise.} \end{cases} \tag{22}$$

The utility function for agent $t \in \mathbb{M}$, $u_t(x,y)$ is a piecewise linear function with kink at ray $y = (2^t - 0.5)x$. The indifference curves have slope of -1 for $y > (2^t - 0.5)x$ and -0.5 for $y < (2^t - 0.5)x$.

[10]The unique common point on the two equality constraints, $x+y=1$ and $x+2y=2$ is $(0,1)$.

[11]Observe that these allocations are not strictly golden rule allocations as discussed in Section 2, as the resource constraint is different.

[12]For illustration, the indifference curves for agent t have same shape for all $x_t > \frac{1}{2^t}$ as the indifference curves of agent $t-1$ for all $x_{t-1} > \frac{1}{2^{t-1}}$. This is useful in showing the envy free nature of the allocations for the histories $x_0 < 1$.

We show that there exists a fair allocation for this economy for all $x_0 \in [0,1]$. Since the stationary allocation $\langle x_t^*, y_{t+1}^* \rangle_{t=0}^{\infty} \equiv \langle 0, 1 \rangle$ is fair for $x_0 = 0$[13], we consider $x_0 \in (0,1]$ only.

We divide the analysis in following two cases.

Case **(a)** $x_0 = 1$: Following allocation is envy free,

$$\langle x_t^*, y_{t+1}^* \rangle_{t=0}^{\infty} \equiv \left\langle \frac{1}{2^t}, 1 - \frac{1}{2^{t+1}} \right\rangle_{t=0}^{\infty}. \tag{23}$$

It is feasible as $x_t^* + y_t^* = \frac{1}{2^t} + 1 - \frac{1}{2^t} = 1$ for all $t \in \mathbb{N}$.

Let set $D \equiv \{(x,y) : (x,y) \in \mathbb{R}_+^2, x + 2y \leqslant 2\}$. Observe that

(i) $\langle x_t^*, y_{t+1}^* \rangle \in D$ for all $t \in \mathbb{M}$ because $x_t^* + 2y_{t+1}^* = \frac{1}{2^t} + 2(1 - \frac{1}{2^{t+1}}) = 2$.

(ii) $\langle x_t^*, y_{t+1}^* \rangle$ is a sequence of optimal allocation for all agent $t \in \mathbb{M}$ over the feasible domain D. The allocation for agent t lies on $x_t + 2 \cdot y_{t+1} = 2$ and is the point of kink on the indifference curve, since,

$$y_{t+1}^* = 1 - \frac{1}{2^{t+1}} = (2^t - 0.5) \cdot \frac{1}{2^t} = (2^t - 0.5) \cdot x_t^*,$$

which yields $u_t(x_t^*, y_{t+1}^*) = 2$.

(iii) All agents $t \in \mathbb{M}$ are indifferent to allocations of all predecessors (if any) and strictly prefer their own allocations compared to allocations of all successors as shown below.

We compare predecessor' allocation first. At any predecessor's allocation, $\langle \frac{1}{2^{t-s}}, 1 - \frac{1}{2^{t-s+1}} \rangle$, for $t > s \geqslant 1$,

$$u_t(x_{t-s}^*, y_{t-s+1}^*) = \frac{1}{2^{t-s}} + 2\left(1 - \frac{1}{2^{t-s+1}}\right) = 2 = u_t(x_t^*, y_{t+1}^*),$$

where we have used the fact that $(\frac{1}{2^t}, 1 - \frac{1}{2^{t+1}})$ is the point of kink on the indifference curve for agent t and $\frac{1}{2^t} < \frac{1}{2^{t-s}}$. For any successor's allocation, $\langle \frac{1}{2^{t+s}}, 1 - \frac{1}{2^{t+s+1}} \rangle$, for $s \geqslant 1$,

$$u_t(x_{t+s}^*, y_{t+s+1}^*) = \left(\frac{2^{t+2}}{1 + 2^{t+1}}\right)\left(\frac{1}{2^{t+s}} + 1 - \frac{1}{2^{t+s+1}}\right) = 2 \cdot \frac{1 + 2^{t+s+1}}{2^s + 2^{t+s+1}} < 2 = u_t(x_t^*, y_{t+1}^*).$$

Hence the allocation is envy free.

(Efficiency) If the allocation is not efficient, then, there exists a feasible Pareto - superior allocation (w.l.o.g. let agent 0 be better off and all other agents be at least indifferent).

[13] It is the unique fair allocation for any set of diverse (which includes identical) preferences satisfying monotonicity axiom (A3).

Thus $y_1 = \left(1-\frac{1}{2}\right)+\varepsilon, \varepsilon > 0$ and $x_1 = \frac{1}{2}-\varepsilon$. To ensure that agent t is at least as well off as at the envy free allocation, his allocation needs to be,

$$x_t = \frac{1}{2^t} - \varepsilon \quad \text{and} \quad y_{t+1} = 1 - \frac{1}{2^{t+1}} + \varepsilon.$$

For all $\varepsilon > 0$, there exists a T such that $\frac{1}{2^T} \geqslant \varepsilon > \frac{1}{2^{T+1}}$. Observe that the allocation for generation T,

$$x_T = \frac{1}{2^T} - \varepsilon \geqslant 0 \quad \text{and} \quad y_{T+1} = 1 - \frac{1}{2^{T+1}} + \varepsilon > 1,$$

is not feasible. Hence the envy free allocation is efficient. We call it *global fair path*.

Case **(b)** $x_0 \in (0,1)$: For all $x_0 \in (0,1)$ there exists a $T \in \mathbb{N}$ such that

$$\frac{T}{2^{T+1}} \leqslant x_0 - \frac{1}{2^T} < \frac{T}{2^T} \tag{24}$$

holds. We denote $x_0 - \frac{1}{2^T}$ by θ. We claim following allocation to be envy free.

$$\langle \hat{x}_t, \hat{y}_{t+1} \rangle_{t=0}^{\infty} \equiv \begin{cases} \left\langle x_0 - \frac{t\cdot\theta}{T}, 1 - x_0 + \frac{(t+1)\cdot\theta}{T} \right\rangle_{t=0}^{T-1}, \\ \left\langle x_t^*, y_{t+1}^* \right\rangle_{t=T}^{\infty}. \end{cases} \tag{25}$$

Agent T and all succeeding generations get same allocation as in case (a) and the preceding generations' allocation lie on the straight line $\hat{x}_t + \hat{y}_{t+1} = 1 + \frac{\theta}{T}$. It is feasible as $\hat{x}_t = x_0 - \frac{t\cdot\theta}{T} < 1$,

$$\begin{aligned} \hat{y}_{t+1} &= 1 - x_0 + \frac{(t+1)\cdot\theta}{T} \leqslant 1 - x_0 + \frac{(T-1+1)\cdot\theta}{T} \\ &= 1 - x_0 + x_0 - \frac{1}{2^T} = 1 - \frac{1}{2^T} < 1, \end{aligned}$$

and $\hat{x}_t + \hat{y}_t = x_0 - \frac{t\cdot\theta}{T} + 1 - x_0 + \frac{t\cdot\theta}{T} = 1$.

We divide the set of agents in two subsets.

(i) Generation $s \geqslant T$: As observed earlier, generation T as well as all its' successors get the same allocation as in case (**a**). Generation s strictly prefers own allocation compared to any of it's successors' allocation as shown in (**a**).

Further, their predecessors' allocation can be further subdivided in two groups.

(A) Some of the predecessors (namely, generations $T, \cdots, s-1$) of generation s get allocation as in (**a**). Generation s is indifferent to allocations of these predecessors as shown, again, in (**a**).

(B) The remaining (generations $0, \cdots, T-1$) predecessors of agent s get allocations given by Eq. (25). Agent s strictly prefer own allocation to the allocations of these predecessors as shown below.

$$\begin{aligned} u_s(\hat{x}_t,\, \hat{y}_{t+1}) &= \hat{x}_t + 2 \cdot \hat{y}_{t+1} \quad = 2 - x_0 + \frac{s+2}{T} \cdot \theta \\ &\leqslant 2 - x_0 + \frac{T-1+2}{T} \cdot \theta \quad = 2 - x_0 + \theta + \frac{\theta}{T} \\ &= 2 - \frac{1}{2^T} + \frac{\theta}{T} \quad < 2 - \frac{1}{2^T} + \frac{T}{T \cdot 2^T} \\ &= 2 \quad = u_s(x_s^*, y_{s+1}^*). \end{aligned}$$

Hence agent s does not envy any other agent's allocation.

(ii) Generation $s \in \{0, \cdots, T-1\}$: Note that for all s, the allocation is such that $\hat{y}_{s+1} > (2^s - 0.5)\hat{x}_s$ as shown below. First, $\hat{x}_s < x_s^*$, for,

$$\begin{aligned} x_s^* - \hat{x}_s &= \frac{1}{2^s} - x_0 + \frac{s \cdot \theta}{T} \geqslant \frac{1}{2^{T-1}} - x_0 + \frac{T-1}{T} \cdot \theta, \\ &= \frac{1}{2^{T-1}} - x_0 + \theta - \frac{\theta}{T} = \frac{1}{2^{T-1}} - \frac{1}{2^T} - \frac{\theta}{T} = \frac{1}{2^T} - \frac{\theta}{T} > 0, \end{aligned}$$

where the last strict inequality follows from Eq (24) and as a consequence of $x_s^* + y_s^* = \hat{x}_s + \hat{y}_s$, we get $\hat{y}_{s+1} \geqslant y_{s+1}^*$ for all $0 \leqslant s \leqslant T-1$. Then

$$\frac{\hat{y}_{s+1}}{\hat{x}_s} \geqslant \frac{y_{s+1}^*}{\hat{x}_s} > \frac{y_{s+1}^*}{x_s^*} = \frac{1 - \frac{1}{2^{s+1}}}{\frac{1}{2^s}} = 2^s - 0.5.$$

Successors of generation s can be divided in two groups.

(C) Generation $T, T+1, \cdots$ who get allocation on the global fair path. Generation s weakly prefers own allocation compared to allocation of these generations as shown below.

$$\begin{aligned} u_s(\hat{x}_s,\, \hat{y}_{s+1}) - u_s(x_t^*, y_{t+1}^*) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right) \cdot (\hat{x}_s + \hat{y}_{s+1}) - \left(\frac{2^{s+2}}{1+2^{s+1}}\right) \cdot (x_t^* + y_{t+1}^*) \\ &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right) \cdot \left(1 + \frac{\theta}{T} - 1 - \frac{1}{2^{t+1}}\right) \\ &\geqslant \left(\frac{2^{s+2}}{1+2^{s+1}}\right) \cdot \left(\frac{1}{T} \cdot \frac{T}{2^{T+1}} - \frac{1}{2^{t+1}}\right) \geqslant 0, \end{aligned}$$

the last weak inequality follows from the fact that $t \geqslant T$.

(D) Generation $s+1,\cdots,T-1$ whose allocation lie on the same straight line $\hat{x}_t+\hat{y}_{t+1}=1+\frac{\theta}{T}$ as generation s. Since $\hat{y}_{t+1}>(2^s-0.5)\cdot\hat{x}_t$ for all such successors,

$$\begin{aligned}u_s(\hat{x}_s,\,\hat{y}_{s+1})-u_s(\hat{x}_t,\,\hat{y}_{t+1}) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\cdot(\hat{x}_s+\hat{y}_{s+1})-\left(\frac{2^{s+2}}{1+2^{s+1}}\right)\cdot(\hat{x}_t+\hat{y}_{t+1})\\ &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\cdot\left(1+\frac{\theta}{T}-1-\frac{\theta}{T}\right) = 0\end{aligned}$$

Thus generation s either weakly prefers (or is indifferent between) own consumption bundle to the allocation of all his successors.

Observe that initial old agent has no predecessor. Predecessors of other agents $s\geqslant 1$ can be divided in two classes.

(E) Generations t for whom $\hat{y}_{t+1}\geqslant(2^s-0.5)\cdot\hat{x}_t$ holds. This case is similar to (D) above. Agent s is indifferent with their allocation as they all lie on the same indifference curve giving utility $\frac{2^{s+2}}{1+2^{s+1}}\cdot\left(1+\frac{1}{T}\left(x_0-\frac{1}{2^T}\right)\right)$.

(F) Generations t for whom $\hat{y}_{t+1}<(2^s-0.5)\cdot\hat{x}_t$ holds. Since $\hat{y}_{s+1}\geqslant(2^s-0.5)\cdot\hat{x}_s$, $u_s(\hat{x}_s,\hat{y}_{s+1})$ is $\frac{2^{s+2}}{1+2^{s+1}}\cdot(1+\frac{\theta}{T})$. The allocation for agent t, $(\hat{x}_t,\hat{y}_{t+1})$ is $(\hat{x}_s+\frac{s-t}{T}\cdot\theta,\hat{y}_{s+1}-\frac{s-t}{T}\cdot\theta)$.

Then,

$$\begin{aligned}u_s(\hat{x}_s,\,\hat{y}_{s+1})-u_s(\hat{x}_t,\,\hat{y}_{t+1}) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\cdot(\hat{x}_s+\hat{y}_{s+1})-(\hat{x}_t+2\cdot\hat{y}_{t+1})\\ &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\cdot\left(1+\frac{\theta}{T}\right)-\left(1+\frac{\theta}{T}+\hat{y}_{t+1}\right)\\ &= \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\cdot\left(1+\frac{\theta}{T}\right)-\hat{y}_{t+1}\\ &= \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\cdot\hat{x}_t-\left(\frac{2}{2^{s+1}+1}\right)\hat{y}_{t+1}\\ &= \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\cdot\hat{x}_t\cdot\left(1-\left(\frac{2}{2^{s+1}-1}\right)\cdot\frac{\hat{y}_{t+1}}{\hat{x}_t}\right) > 0,\end{aligned}$$

where in the last strict inequality, we have used the fact that $\frac{\hat{y}_{t+1}}{\hat{x}_t}<(2^s-0.5)$. Thus agent s strictly prefers own consumption bundle compared to assignment of all such predecessors.

Hence, in all cases, agent s weakly prefers his own allocation to all others' allocation. The allocation in Eq. (25) is therefore envy free.

The efficiency of this allocation follows from the fact that it is on global fair path for generations $s\geqslant T$ and there is no wastage of resources in the allocations for the first $T-1$ generations.

5 Stationary OLG Economy with agents alive for three periods

In this section we consider overlapping generations economy with each agent living for three periods, young, working and retired. In this case, the consumption bundle of agents will be denoted by (x_t, y_{t+1}, z_{t+2}) where x_t, y_{t+1} and z_{t+2} are young age, working age and retired age consumptions for agent t, respectively. The set of consumption bundles will now be $X \equiv \mathbb{R}^3_+$.

In period two, agent 2 is born, agent 1 is working and agent 0 is retired. This ensures overlap among all three generations of agents in each period. A history for this economy would consist of three scalars, namely $\{x_0, y_1\}$, the young and working age consumption of initial retired agent and x_1, the young age consumption of initial working agent.

An *allocation* is a sequence $\langle x_t, y_{t+1}, z_{t+2}\rangle = \langle x_t, y_{t+1}, z_{t+2}\rangle_{t=0}^{\infty}$ satisfying,

$$\left.\begin{array}{lcl} (x_t, y_{t+1}, z_{t+2}) & \geqslant & 0 \quad \text{for all } t \in \mathbb{M}, \\ x_t + y_t + z_t & \leqslant & 1 \quad \text{for all } t > 1, \\ x_0 = \bar{x}_0,\ y_1 = \bar{y}_1 & \text{and} & x_1 = \bar{x}_1. \end{array}\right\} \tag{26}$$

An allocation $\langle x_t, y_{t+1}, z_{t+2}\rangle_{t=0}^{\infty}$ is *stationary* if,

$$(x_t, y_{t+1}, z_{t+2}) = (x_{t+1}, y_{t+2}, z_{t+3}) \ \text{ for all } t \in \mathbb{M}. \tag{27}$$

Observe that for a stationary allocation to exist, we need $\bar{x}_0 = \bar{x}_1$.

We continue to assume that the preferences of agents are nicely behaved, in the sense that they are represented by a utility function $u : X \to \mathbb{R}$ which satisfies $A1$, $A2$ and $A3$.

5.1 Golden Rule

As in the case of two period OLG economy, a golden-rule is a triple $(\bar{x}, \bar{y}, \bar{z})$ which solves the following problem

$$\left.\begin{array}{ll} \max & u(x,y,z) \\ \text{subject to} & x+y+z \leqslant 1 \\ \text{and} & (x,y,z) \geqslant 0. \end{array}\right\} \tag{GR1}$$

In words, for any historically given $\bar{x}_0 = \bar{x}_0 \in [0,1]$ and $\bar{y}_1 \in [0, 1-\bar{x}_1]$, consider the stationary allocation $\langle x, y, z\rangle$ which does not waste any endowment. This is given by $\langle \bar{x}_0, \bar{y}_1, 1 - \bar{x}_0 - \bar{y}_1\rangle$ with stationary utility of $u(\bar{x}_0, \bar{y}_1, 1-\bar{x}_0-\bar{y}_1)$. Now, consider all economies possible with different historically given levels of $\bar{x}_0 = \bar{x}_0 \in [0,1]$ and $\bar{y}_0 \in [0, 1-\bar{x}_0]$. Pick the economy which has the largest utility level $u(\bar{x}_0, \bar{y}_1, 1-\bar{x}_0-\bar{y}_1)$ among all such economies. This corresponds to the golden-rule.

We will again assume that u is such that,

A5. There is a unique solution $(\bar{x},\bar{y},\bar{z})$ to problem (GR1), and $(\bar{x},\bar{y},\bar{z}) >> 0$.

It follows, of course, from A3 and A5 that

$$\bar{x}+\bar{y}+\bar{z}=1. \tag{28}$$

Let us define the sets

$$\begin{aligned} C &= \{(x,y,z)\in[0,1]\times[0,1]\times[0,1] : u(x,y,z)\geqslant u(\bar{x},\bar{y}),\bar{z}\},\\ D &= \{(x,y,z)\in[0,1]\times[0,1]\times[0,1] : u(x,y,z)>u(\bar{x},\bar{y},\bar{z})\}. \end{aligned} \tag{29}$$

It follows from the definition of a golden-rule that

$$x+y+z>1 \text{ for all } (x,y,z)\in D, \tag{30}$$

and it follows from the definition of a golden-rule and A3 that

$$x+y+z\geqslant 1 \text{ for all } (x,y,z)\in C. \tag{31}$$

Following lemma is equivalent to the Lemma 1 for two periods living agents economy. We skip the proof.

Lemma 3. *Given any* $\varepsilon>0$*, there is* $\delta>0$ *such that whenever* $(x,y,z)\in C(\varepsilon)$*, where*

$$C(\varepsilon) = \{(x,y,z)\in[0,1]\times[0,1]\times[0,1] : u(x,y,z)\geqslant u(\bar{x},\bar{y},\bar{z})+\varepsilon\},$$

we must have

$$x+y+z\geqslant 1+\delta \tag{32}$$

Proof. Omitted. □

5.2 Sufficient conditions for no fair allocation

5.2.1 No envy free allocation

Lemma 4. *Suppose* $\{\bar{x}_0,\bar{y}_1\}$ *are such that* $u(\bar{x}_0,\bar{y}_1,0)>u(\bar{x},\bar{y},\bar{z})$ *holds. Then, there is no fair allocation.*

Proof. For an allocation to be envy-free, we must have $u(x_t,y_{t+1},z_{t+2})$ constant for all $t\in\mathbb{M}$. Denote this constant utility level by v. We claim that $v\leqslant u(\bar{x},\bar{y},\bar{z})$. For if $v>u(\bar{x},\bar{y},\bar{z})$, then denoting $v-u(\bar{x},\bar{y},\bar{z})$ by ε, we have $\varepsilon>0$, and by Lemma 3, there is $\delta>0$, such that,

$$x_t+y_{t+1}+z_{t+2}\geqslant 1+\delta=\bar{x}+\bar{y}+\bar{z}+\delta \text{ for all } t\in\mathbb{M}.$$

Noting that $x_t + y_t + z_t \leqslant 1$ for all $t \geq 2$, we then obtain,

$$\begin{aligned} z_2 - \bar{z} &\geqslant (\bar{x} - \bar{x}_0) + (\bar{y} - \bar{y}_1) + \delta \\ z_3 - \bar{z} &\geqslant (x_2 - \bar{x}_1) + (z_2 - \bar{z}) + \delta \\ &\ldots\ldots \\ z_{t+2} - \bar{z} &\geqslant (x_{t+1} - x_t) + (z_{t+1} - \bar{z}) + \delta. \end{aligned}$$

This gives

$$z_{t+2} - \bar{z} \geqslant (\bar{x} - \bar{x}_0) + (x_{t+1} - \bar{x}_1) + (\bar{y} - \bar{y}_1) + (t+1) \cdot \delta \tag{33}$$

But, this implies that $(z_{t+2} - \bar{z}) \to \infty$ as $t \to \infty$, which contradicts the fact that $z_{t+2} \leqslant 1$ for all $t \in \mathbb{M}$. This establishes the claim.

Then, $u(\bar{x}_0, \bar{y}_1, z_2) \geq u(\bar{x}_0, \bar{y}_1, 0) > v$ is a contradiction.

□

5.2.2 No envy free allocation is fair

We are now in a position to state a sufficient condition for the parameter values for which there does not exist any fair allocation in the stationary case.

Proposition 1. *Suppose $\{\bar{x}_0, \bar{y}_1, \bar{x}_1\}$ are such that for some natural number, $1 < T < \infty$ following conditions hold:*

There exist $\{\hat{x}_t, \hat{y}_t, \hat{z}_t\} > 0$, $\hat{x}_t + \hat{y}_t + \hat{z}_t \leq 1$ for all $t \leq T$; $\{\hat{y}_{T+1}, \hat{z}_{T+1}\} > 0$ and $\hat{y}_{T+1} + \hat{z}_{T+1} \leq \bar{y} + \bar{z}$, such that

$$\{u(\bar{x}_0, \bar{y}_1, \hat{z}_2), u(\bar{x}_1, \hat{y}_2, \hat{z}_3) \cdots, u(\bar{x}_T, \hat{y}_{T+1}, \bar{z})\} > \{u(\bar{x}, \bar{y}, \bar{z}), \cdots, u(\bar{x}, \bar{y}, \bar{z})\} \tag{34}$$

holds. Then, there is no fair allocation.

Proof. Suppose, on the contrary, there is a fair allocation, call it $\langle x_t, y_{t+1}, z_{t+2} \rangle$. Since it is envy-free, we must have $u(x_t, y_{t+1}, z_{t+2})$ constant for all $t \in \mathbb{M}$. Denote this constant utility level by v. In Proposition 1, we have shown that $v \leqslant u(\bar{x}, \bar{y}, \bar{z})$.

Consider now the sequence $\langle x'_t, y'_{t+1}, z'_{t+2} \rangle$ defined by

$$\begin{aligned} &x'_0 = \bar{x}_0, x'_1 = \bar{x}_1, y'_1 = \bar{y}_1 \\ &(x'_t, y'_t, z'_t) = (\hat{x}_t, \hat{y}_t, \hat{z}_t), \text{ for all } 1 < t \leq T \\ &(x'_{T+1}, y'_{T+1}, z'_{T+1}) = (\bar{x}, \hat{y}_{T+1}, \hat{z}_{T+1}); \text{ and} \\ &(x'_t, y'_t, z'_t) = (\bar{x}, \bar{y}, \bar{z}) \text{ for all } t \geq T+2. \end{aligned}$$

It is easy to check then that $\langle x'_t, y'_{t+1}, z'_{t+2}\rangle$ is an allocation. Further,

$$u(x'_t, y'_{t+1}, z'_{t+2}) = u(\bar{x}, \bar{y}, \bar{z}) = v \geqslant u(x_t, y_{t+1}, z_{t+2}) \text{ for all } t \geq T+2$$

and

$$(u(x'_0, y'_1, z'_2), \cdots, u(x'_{T+1}, y'_{T+2}, z'_{T+3})) > (u(\bar{x}, \bar{y}, \bar{z}), \cdots, u(\bar{x}, \bar{y}, \bar{z})) = (v, \cdots, v).$$

establishing that $\langle x_t, y_{t+1}, z_{t+2}\rangle$ is inefficient. This contradiction establishes the result. □

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