

On Monotone Social Welfare Orders satisfying the Strong Equity Axiom: Construction and Representation

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Abstract

This paper studies the nature of social welfare orders on infinite utility streams, satisfying the efficiency principle known as Monotonicity and the consequentialist equity principle known as Strong Equity. It provides a complete characterization of domain sets for which there exists such a social welfare order which is in addition representable by a real valued function. It then shows that for those domain sets for which there is no such social welfare order which is representable, the existence of such a social welfare order necessarily entails the existence of a non-Ramsey set, a non-constructive object.

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1 Introduction

The conflict between principles of equity and efficiency in the evaluation of infinite utility streams¹ has been discussed extensively in the literature.

The analysis of such conflicts depends, of course, on the precise nature of the efficiency and equity principles that are imposed. In this paper, we focus on the weakest efficiency principle, which is generally referred to as monotonicity (M). This efficiency concept is incontrovertible as it only requires that if no one is worse off (in utility stream x compared to y), then the society as a whole should not be worse off (in utility stream x compared to y).²

The equity concept that we examine in this paper (called the Strong Equity axiom) belongs to the class of *consequentialist* equity concepts, dealing with situations in which the *distribution* of utilities of generations has changed in specific ways. The Strong Equity (SE) axiom is a strong form of the equity axiom of Hammond (1976)³ and involves comparisons between two utility streams (x and y) in which all generations except two have the same utility levels in both utility streams. Regarding the two remaining generations (say, i and j), one of the generations (say i) is better off in utility stream x , and the other generation (j) is better off in utility stream y , thereby setting up a conflict. The axiom states that if for both utility streams, it is generation i which is worse off than generation j (this, of course, requires us to make intergenerational comparisons of utilities), then generation i should be allowed (on

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¹We use the standard framework in which the space of infinite utility streams is given by $X = Y^{\mathbb{N}}$, where Y is a non-empty set of real numbers, and \mathbb{N} the set of natural numbers.

²See Diamond (1965, p.172), and Basu and Mitra (2007).

³We use the term “strong” relative to the axiom introduced by Hammond (1976), which he called the *Equity Axiom*, and which (as he explains) is in the spirit of the *Weak Equity Axiom* of Sen (1973). The SE axiom was introduced by d’Aspremont and Gevers (1977), who referred to it as an *Extremist Equity Axiom*.

behalf of the society) to choose between x and y . That is, x is socially preferred to y , since generation i is better off in x than in y .

The equity axiom of Hammond is one of the key consequentialist equity concepts, the other being the Pigou-Dalton transfer principle.⁴ Note that under the situation described in the previous paragraph, Hammond Equity would make the weaker statement that x is at least as good as y . Since our efficiency concept (Monotonicity) does not require sensitivity, the combination of Hammond Equity and Monotonicity would clearly be satisfied by the trivial social welfare order which considers all utility streams to be socially indifferent. Thus, having accepted Monotonicity as the incontrovertible efficiency concept, it is natural to focus on SE, the stronger form of the equity axiom of Hammond, and to ask whether these two axioms are compatible.

Bossert, Sprumont and Suzumura (2007) have shown that there exist social welfare orders on infinite utility streams which satisfy Hammond Equity and Strong Pareto.⁵ Under Strong Pareto, Hammond Equity and Strong Equity are equivalent. So, in particular, we know that there exist social welfare orders on infinite utility streams which satisfy the Strong Equity Axiom and Monotonicity. Our paper is devoted to understanding the nature of such social welfare orders.

Clearly such orders can be useful in decision making provided they can be *represented* by a real-valued function, or at least (even if they do not have a real-valued representation) if they can be *constructed*.

The existence of social welfare orders, satisfying Hammond Equity and Strong Pareto, is established in Bossert et. al. (2007) by using the variant of Szpilrajn's Lemma given in Arrow (1951), a non-constructive device. This, of course, leaves open the question of whether such social welfare orders can be constructed. More pertinent from the standpoint of the current investigation, it leaves open the question of whether social welfare orders, satisfying the Strong Equity axiom and Monotonicity, can be constructed.

Turning to the representation issue, it is known (see Alcantud and Garcia-Sanz (2010)) that any social welfare order satisfying Hammond Equity and Strong Pareto cannot be represented by a real-valued function, if the domain set (Y) consists of at least four distinct elements. That is, an impossibility result arises as soon as we admit a situation in which Hammond Equity can play a role in ranking two utility streams. In particular, then, the same impossibility result arises for any social welfare order satisfying Strong Equity and Strong Pareto. But, this leaves open the question of whether social welfare orders, satisfying the Strong Equity axiom and Monotonicity, can be represented.

In this paper, we settle both the open questions stated in the previous two paragraphs. Tackling the representation issue first, we show that there exist social welfare functions satisfying the Strong Equity axiom and Monotonicity if and only if the domain set (Y) has at most five distinct elements.⁶ We would especially like to emphasize the possibility result contained in this characterization, since there are so few possibility results in this literature. Previous results would tend to suggest that an impossibility result would hold as soon as we admit a situation in which Strong Equity can play a role in ranking two utility streams; that is, as soon as we admit four distinct elements in the domain set Y . That this

⁴Hammond Equity has several variations which have been discussed in the literature. Strong Equity and *Hammond Equity for the Future* (see Asheim, Mitra and Tungodden (2007) and Banerjee (2006)) are notable variations; others are of minor importance conceptually.

Altruistic Equity is a variation of the Pigou-Dalton transfer principle (see Hara et.al. (2008)). We note that an analysis similar to that conducted in our paper can be carried out when the Strong Equity axiom is replaced by the Pigou-Dalton transfer principle or Altruistic Equity.

⁵Hammond Equity is called "Equity Preference" in their paper. The social welfare orders obtained by them also satisfy Finite Anonymity, a notion of *procedural equity*.

⁶Since establishing this complete characterization, we have become aware that the impossibility part of the characterization is similar to a result established by Alcantud (2011), using an equity concept slightly stronger than SE, when Y has seven or more distinct elements.

is *not* so is quite surprising. Further, if one thinks of the equity concept involved in Hammond Equity or in Strong Equity, it is difficult to see how a social welfare function can be obtained on infinite utility streams, since the pair of generations affected might be located at *any two* points in the infinite horizon. Our possibility result is established by an *explicit construction* of the social welfare function (when Y consists of at most five distinct elements), which is of considerable interest.

The low cardinality requirement (on the set Y) for our possibility result should not obscure the fact that the social welfare function is defined on X , which still consists of an *uncountably infinite* number of distinct utility streams. Further, from the practical policy point of view, we do quite often deal with a low cardinality of the domain set Y , using distinct utility levels to distinguish (for instance) the “rich”, the “poor”, the “upper middle class”, and the “lower middle class”.

Turning to the construction issue, the question we address is the following. When the domain set (Y) has more than five distinct elements (a situation in which we know that there is no social welfare function satisfying the Strong Equity axiom and Monotonicity) is it possible to construct a social welfare order satisfying the Strong Equity axiom and Monotonicity? We show, using a variation of the method introduced by Lauwers (2010), that when the domain set (Y) has more than five distinct elements, the existence of any social welfare order satisfying the Strong Equity axiom and Monotonicity implies the existence of a non-Ramsey set, a non-constructive object.

In this context, then, if there is no representable social welfare order satisfying the Strong Equity axiom and Monotonicity, then there is no social welfare order satisfying the Strong Equity axiom and Monotonicity which can be constructed.⁷

2 Preliminaries

2.1 Notation

Let \mathbb{R} and \mathbb{N} be the sets of real numbers and natural numbers respectively. Let Y , a non-empty subset of \mathbb{R} , be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^{\mathbb{N}}$ is the set of all possible utility streams. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \dots)$, where, for all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that the generation of period n earns.

Let T be an infinite subset of \mathbb{N} . We denote by $\Omega(T)$ the collection of all infinite subsets of T , and we denote $\Omega(\mathbb{N})$ by Ω . Thus, for any infinite subset T of \mathbb{N} , we have $T \subset \mathbb{N}$, and $T \in \Omega$.

For all $y, z \in X$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.2 Definitions

2.2.1 Social Welfare Order, Efficiency and Equity

We consider binary relations on X , denoted by \succsim , with symmetric and asymmetric parts denoted by \sim and \succ respectively, defined in the usual way. A *social welfare order* (SWO) is a complete and transitive binary relation.

A *social welfare function* (SWF) is a mapping $W : X \rightarrow \mathbb{R}$. Given a SWO \succsim on X , we say that \succsim can be *represented* by a real-valued function if there is a mapping $W : X \rightarrow \mathbb{R}$ such that for all $x, y \in X$, we have $x \succsim y$ if and only if $W(x) \geq W(y)$.

⁷A similar result, with respect to the efficiency concept of Weak Pareto and the procedural equity concept of Finite Anonymity, is shown to hold in Dubey (2011), after establishing the conjecture of Fleurbaey and Michel (2003).

The social welfare orders that we will be concerned with are required to satisfy an efficiency axiom and an equity axiom. Our efficiency requirement is very weak, and it is called Monotonicity. It is difficult to consider a social welfare order seriously if it violates this axiom.

Definition 1 Monotonicity (M): *If $x, y \in X$, with $x \geq y$, then $x \succsim y$.*

The equity requirement that we use is called Strong Equity; the rationale for it has already been explained in Section 1.

Definition 2 Strong Equity (SE): *If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.*

It is convenient to define analogous concepts pertaining to a social welfare function $W : X \rightarrow \mathbb{R}$.

Definition 3 W-Monotonicity: *If $x, y \in X$, with $x \geq y$, then $W(x) \geq W(y)$.*

Definition 4 W-Strong Equity (SE): *If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $W(x) > W(y)$.*

2.2.2 Ramsey Collection of Sets

A collection of sets $\Gamma \subset \Omega$ is called *Ramsey* if there exists $T \in \Omega$ such that either $\Omega(T) \subset \Gamma$ or $\Omega(T) \subset \Omega \setminus \Gamma$.

One can consider Ω to be a topological space, endowed with the standard product topology. Galvin and Prikry (1973) showed that if $\Gamma \subset \Omega$ is any Borel set, then it is Ramsey; in particular, if $\Gamma \subset \Omega$ is any open set, then it is Ramsey. Generalizing the Galvin-Prikry result, Silver (1970) has shown that if $\Gamma \subset \Omega$ is any analytic set, then it is Ramsey.

On the other hand, not every collection of sets $\Gamma \subset \Omega$ is Ramsey. Erdős and Rado (1952, Example 1, p. 434) have shown, using Zermelo's well-ordering principle (which is known to be equivalent to the Axiom of Choice), that there is a collection of sets $\Gamma \subset \Omega$, such that for every $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and its complement $\Omega \setminus \Gamma$. Such a collection of sets $\Gamma \subset \Omega$ is called *non-Ramsey*.

3 Results

In this section, we present our results on representation and construction of SWOs satisfying the Monotonicity and Strong Equity axioms. Proofs of the results are presented in Section 4.

3.1 Representation

First, we construct explicitly a social welfare function satisfying the Strong Equity and Monotonicity axioms when the domain set Y consists of a set of five distinct real numbers.

Proposition 1 *Let $Y \equiv \{a, b, c, d, e\}$ be such that $a < b < c < d < e$ and let $X \equiv Y^{\mathbb{N}}$. Given any sequence $x \in X$, define:*

$$N(x) = \{n : n \in \mathbb{N} \text{ and } x_n = a\}, \text{ and } M(x) = \{m : m \in \mathbb{N} \text{ and } x_m = b\}.$$

Let $\alpha(n) = -(1/2^n)$, $\beta(n) = -(1/3^n)$ and $\delta(n) = -\alpha(n)$ for all $n \in \mathbb{N}$. Define the social welfare function $W : X \rightarrow \mathbb{R}$ by

$$W(x) = \begin{cases} \sum_{n \in N(x)} \alpha(n) + \sum_{m \in M(x)} \beta(m) & \text{if } N(x) \text{ or } M(x) \text{ is non-empty} \\ \sum_{n=1}^{\infty} \delta(n)(x_n - a) & \text{otherwise.} \end{cases} \quad (1)$$

Then, $W(x)$ satisfies W -Strong Equity and W -Monotonicity.⁸

Remark:

When the domain set Y contains four distinct elements, $Y \equiv \{a, b, c, d\}$ where $a < b < c < d$, the function:

$$W(x) = \begin{cases} \sum_{n \in N(x)} \alpha(n) & \text{if } N(x) \text{ is non-empty} \\ \sum_{n=1}^{\infty} \delta(n)(x_n - a) & \text{otherwise} \end{cases} \quad (2)$$

(with notation as in Proposition 1 above) can be shown to satisfy both W -Strong Equity and W -Monotonicity, by using a proof similar to the proof of Proposition 1.

We can explain informally the nature of the social welfare function W in (2) above, which is proposed in the case in which there are four distinct elements in Y .⁹ The key idea is to separate the “poor” state a from the richer states $\{b, c, d\}$, making W entirely insensitive to the utilities of the richer generations whenever there is even a single poor generation, and fully sensitive to the presence of each poor generation. It is a standard discounted sum of utilities when there is no generation in the poor state.

In the situation that strong equity needs to be verified, we have alternatives $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. In this case, we can infer that y_i must be equal to a , and so alternative y has at least one poor generation, namely i . We can also infer that neither generation i nor j is poor in alternative x . Thus, if there is some generation $k \neq i, j$ for which generation k is poor in alternative x , the first formula in (2) applies to both x and y , and $W(x) > W(y)$ because y has one more poor generation than x . On the other hand, if there is no such k , then the first formula in (2) applies to y , while the second formula in (2) applies to x , and again $W(x) > W(y)$, since $W(x) > 0$ while $W(y) < 0$.

In the situation that monotonicity needs to be verified, we have alternatives $x, y \in X$, and $x \geq y$. If both x and y have a poor generation, then the first formula in (2) applies to both x and y , and so $W(x) \geq W(y)$.¹⁰ If neither x nor y has a poor generation, then the second formula in (2) applies to both x and y , and so again $W(x) \geq W(y)$. There is, of course, a third possibility: y has a poor generation but x does not. In this case, the first formula in (2) applies to y , while the second formula in (2) applies to x , and so we have $W(x) > 0 > W(y)$.

Combining Proposition 1 and the above remark with an impossibility result when Y has more than five distinct elements, we obtain the following complete characterization result on representable SWOs.

Theorem 1 *There exists a representable social welfare order on $X \equiv Y^{\mathbb{N}}$, satisfying the Monotonicity and Strong Equity axioms, if and only if Y consists of at most five distinct elements.*

⁸It can be checked that W , defined by (1), also satisfies W -Weak Pareto; that is, if $x, y \in X$, and $x \gg y$, then $W(x) > W(y)$.

⁹The explanation of the nature of the social welfare function W in (1) above, which is proposed in the case in which there are five distinct elements in Y , is similar but more subtle, since there are two relatively poor states, and both can be active states in a strong equity comparison.

¹⁰Note that in this case, the precise utility information about the generations which are not poor is totally ignored. Since our efficiency concept is Monotonicity, this insensitivity in W does not create any problem.

3.2 Construction

We will say that a collection of sets $\Gamma \subset \Omega$ can be *constructed* if it can be obtained in *every* model of set theory, satisfying the Zermelo-Frankel axioms. If a collection of sets $\Gamma \subset \Omega$ cannot be constructed, we call it a *non-constructive* object.

In a seminal paper, Solovay (1970) obtained a model of set theory, satisfying the Zermelo-Frankel axioms, in which the Axiom of Choice is false and every set of reals is Lebesgue measurable.¹¹ Subsequently, Mathias (1977) showed that in Solovay's model every collection of sets $\Gamma \subset \Omega$ is Ramsey. This implies that any collection of sets $\Gamma \subset \Omega$ which is non-Ramsey cannot be obtained in Solovay's model and so, according to our definition, cannot be constructed.¹² Thus, a non-Ramsey collection of sets $\Gamma \subset \Omega$ is a non-constructive object.

Our principal result is that when the domain set (Y) has more than five distinct elements, the existence of any social welfare order satisfying the Strong Equity axiom and Monotonicity implies the existence of a non-Ramsey set, a non-constructive object.

Theorem 2 *If Y consists of more than five distinct elements, and there is a social welfare order on $X \equiv Y^{\mathbb{N}}$, which satisfies the Monotonicity and Strong Equity axioms, then there is a collection of sets $\Gamma \subset \Omega$, which is non-Ramsey.*

We try to explain informally the content of Theorem 2. To this end, observe that a social welfare order is concerned with ranking of utility streams in $X = Y^{\mathbb{N}}$, and Ramsey or non-Ramsey refers to collections of infinite subsets of \mathbb{N} ; so we need to link the two. One might do this by adopting *rules* which determine the assignments of utilities to the various generations, depending on how the set of all generations is partitioned. Thus, any such rule would be a function from an infinite subset N of \mathbb{N} to X .

Let us say that we have adopted two such rules. Given any infinite subset N of \mathbb{N} , the first rule determines an assignment $x(N)$ in X , and the second rule determines an assignment $y(N)$ in X . The social welfare order \succsim will be able to compare any such pair of assignments (that is, rank $x(N)$ relative to $y(N)$).

Now, suppose that we have an arbitrary partition of generations; that is, an infinite subset N of \mathbb{N} . We can then consider an infinite subset T of N (that is, a coarser partition of generations in which some of the points of the original partition are dropped). And, just as we defined assignment $x(N)$ by the first rule and $y(N)$ by the second rule, we can define assignment $x(T)$ by the first rule and $y(T)$ by the second rule. One can now enquire whether the \succsim ranking of $x(N)$ relative to $y(N)$ is preserved or reversed when we compare instead $x(T)$ to $y(T)$.

What Theorem 2 effectively establishes is that we can set up two rules of assignments of utilities to the various generations (depending on how the set of all generations is partitioned) such that for *every* (infinite) partition N of generations, there is *some* coarser (infinite) partition T (of N) of generations, such that the ranking of $x(T)$ relative to $y(T)$ *differs* from the ranking of $x(N)$ relative to $y(N)$.

4 Proofs

Proof (of Proposition 1). We first take up W -Strong Equity. Let $x, y \in X$, and let there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. There are three cases to consider.

¹¹Solovay's model also satisfies the Axiom of Dependent Choice, which in turn implies the Axiom of Denumerable Choice.

¹²In particular, the non-Ramsey collection of sets obtained by Erdős and Rado (1952) cannot be constructed.

(a) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that (i) $y_i = a < b = x_i < x_j = c < d = y_j$ or $y_i = a < b = x_i < x_j = c < e = y_j$ or $y_i = a < b = x_i < x_j = d < e = y_j$ and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. We need to show that $W(x) > W(y)$. Note that $N(x) \cup \{i\} = N(y)$ and $M(x) = M(y) \cup \{i\}$. Then,

$$\begin{aligned} W(x) &= \sum_{n \in N(x)} \alpha(n) + \sum_{n \in M(x)} \beta(n) = \sum_{n \in N(x)} \alpha(n) + \sum_{n \in M(y)} \beta(n) + \beta(i), \\ W(y) &= \sum_{n \in N(y)} \alpha(n) + \sum_{n \in M(y)} \beta(n) = \sum_{n \in N(x)} \alpha(n) + \alpha(i) + \sum_{n \in M(y)} \beta(n), \text{ and} \\ W(x) - W(y) &= \beta(i) - \alpha(i) = \beta(i) + \delta(i) > 0. \end{aligned}$$

(b) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that (i) $y_i = a < c = x_i < x_j = d < e = y_j$ and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. Note that $N(x) \cup \{i\} = N(y)$ and $M(x) = M(y)$. For $\langle x \rangle$, the following sub-cases arise: **(i)** $N(x)$ or $M(x)$ are non-empty; **(ii)** both $N(x)$ and $M(x)$ are empty.

In subcase **(i)**,

$$\begin{aligned} W(x) &= \sum_{n \in N(x)} \alpha(n) + \sum_{n \in M(x)} \beta(n), \\ W(y) &= \sum_{n \in N(y)} \alpha(n) + \sum_{n \in M(y)} \beta(n) = \sum_{n \in N(x)} \alpha(n) + \alpha(i) + \sum_{n \in M(y)} \beta(n) < 0, \text{ and} \\ W(x) - W(y) &= -\alpha(i) = \delta(i) > 0. \end{aligned}$$

In subcase **(ii)**, $W(x) = \sum_{n=1}^{\infty} \delta(n)(x_n - a) > 0 > W(y)$.

(c) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that (i) $y_i = b < c = x_i < x_j = d < e = y_j$ and (ii) $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. Note that $N(x) = N(y)$ and $M(x) \cup \{i\} = M(y)$. For $\langle x \rangle$, the following sub-cases arise: **(i)** $N(x)$ or $M(x)$ are non-empty; **(ii)** both $N(x)$ and $M(x)$ are empty.

In subcase **(i)**, we have:

$$\begin{aligned} W(x) &= \sum_{n \in N(x)} \alpha(n) + \sum_{n \in M(x)} \beta(n), \\ W(y) &= \sum_{n \in N(y)} \alpha(n) + \sum_{n \in M(y)} \beta(n) = \sum_{n \in N(x)} \alpha(n) + \sum_{n \in M(x)} \beta(n) + \beta(i) < 0, \text{ and} \\ W(x) - W(y) &= -\beta(i) > 0. \end{aligned}$$

In subcase **(ii)**, $W(x) = \sum_{n=1}^{\infty} \delta(n)(x_n - a) > 0 > W(y)$.

Next, we establish W -Monotonicity. Let $x, y \in X$ with $x \geq y$. There are again three cases to consider.

(a) Let $y_n > b$ for all $n \in \mathbb{N}$. In this case, $N(x) = M(x) = N(y) = M(y) = \emptyset$. Then, $W(\cdot)$, being sum of discounted one period utilities, clearly satisfies W -Monotonicity.

(b) Let $y_n > a$ for all $n \in \mathbb{N}$ and $y_n = b$ for some n . In this case, the sets $N(x) = N(y) = \emptyset$ but $M(y)$ and possibly $M(x)$ are non-empty. Observe that $M(x) \subset M(y)$. If $M(x) = \emptyset$, then $W(x) > 0 > W(y)$. If $M(x) \neq \emptyset$, let $N_1 \equiv M(y) \setminus M(x)$. Then,

$$W(x) - W(y) = - \sum_{n \in N_1} \beta(n) \geq 0.$$

(c) Let $y_n = a$ for some $n \in \mathbb{N}$. There are two sub-cases to consider: (i) $x_n > b$ for all $n \in \mathbb{N}$; (ii) $x_n \leq b$ for some $n \in \mathbb{N}$.

In subcase (i), $W(x) > 0 > W(y)$, so W -Monotonicity is clearly satisfied.

In subcase (ii), define $\bar{N} \equiv \{n \in \mathbb{N} : x_n \neq y_n\}$, $N_2 \equiv \{n \in \bar{N} : y_n = a; x_n = b\}$, $N_3 \equiv \{n \in \bar{N} : y_n = a; x_n > b\}$, and $N_4 \equiv \{n \in \bar{N} : y_n = b\}$. Then,

$$W(x) - W(y) = \sum_{n \in N_2} (-\alpha(n) + \beta(n)) - \sum_{n \in N_3} \alpha(n) - \sum_{n \in N_4} \beta(n) \geq 0.$$

establishing W -Monotonicity. ■

Proof (of Theorem 1). (**Sufficiency**) Using Proposition 1, and the remark following it, the social welfare function $W : X \rightarrow \mathbb{R}$ (defined by (1) and (2) respectively) satisfies W -Monotonicity and W -Strong Equity when Y has five (four respectively) distinct elements. Now, define \succsim on X by:

$$\text{For all } x, y \in X, x \succsim y \text{ if and only if } W(x) \geq W(y)$$

Then, \succsim is a social welfare order, representable by W , which satisfies Monotonicity and Strong Equity.

(**Necessity**) Suppose, on the contrary, there is $Y = \{a, b, c, d, e, f\}$, where $a < b < c < d < e < f$, and \succsim is a representable social welfare order on $X = Y^{\mathbb{N}}$ satisfying Monotonicity and Strong Equity. Let $W : X \rightarrow \mathbb{R}$ be a function which represents \succsim on X .

The proof is a variation of the method introduced by Basu and Mitra (2003). For each real number $p \in I$, define,

$$N(p) = \{n \in \mathbb{N}; n > 2 : r_n \in (0, p)\} \text{ and } M(p) = \mathbb{N} \setminus \{N(p) \cup \{1, 2\}\}.$$

Define following pair of sequences $\langle x(p) \rangle$ and $\langle y(p) \rangle$ with $x, y \in X$ as follows,

$$x_n(p) = \begin{cases} f & \text{if } n = 1 \\ e & \text{if } n \in N(p) \\ c & \text{if } n = 2 \\ a & \text{otherwise} \end{cases} \quad (3)$$

and

$$y_n(p) = \begin{cases} e & \text{if } n = 1 \\ e & \text{if } n \in N(p) \\ d & \text{if } n = 2 \\ a & \text{otherwise} \end{cases} \quad (4)$$

Note that the only elements of sequence $y(p)$, different from $x(p)$, are $x_2(p) = c < d = y_2(p) < y_1(p) = e < f = x_1(p)$. Hence by Strong Equity, $y(p) \succ x(p)$, and $W(y(p)) > W(x(p))$.

Now let $q \in (p, 1)$. Observe that $N(p) \subset N(q)$ and $M(q) \subset M(p)$. There are infinitely many elements in $N(q) \cap M(p)$. Let $j(p, q) \equiv \min\{N(q) \cap M(p)\}$ for which $y_{j(p, q)}(p) = a < e = x_{j(p, q)}(q)$ holds.

We construct sequence $x'(q)$ from $x(q)$ by changing the $j(p, q)$ element of it as follows: $x'_{j(p, q)}(q) = b < e = x_{j(p, q)}(q)$. Then $y_{j(p, q)}(p) = a < b = x'_{j(p, q)}(q) < c = x'_2(q) < d = y_2(p)$; and $x'_n(q) \geq y_n(p)$ for all other $n \in \mathbb{N}$. This implies $x'(q) \succ y(p)$ by Strong Equity and Monotonicity. Also $x(q) \succsim x'(q)$ by Monotonicity. Therefore, $x(q) \succ y(p)$ by transitivity of \succsim , and so $W(x(q)) > W(y(p))$.

For any given $p \in (0, 1)$, using the function W , we construct an interval $I(p) = (W(x(p)), W(y(p)))$. Then $I(q)$ lies entirely to the right of $I(p)$ and $I(p) \cap I(q) = \emptyset$.

In this manner, we are able to map uncountably many real numbers in $(0, 1)$ into disjoint intervals on the real line which can be at most countable. This contradiction establishes the result. ■

Proof (of Theorem 2). Define $Y \equiv \{a, b, c, d, e, f\}$, with $a < b < c < d < e < f$ and define a pair of utility streams from any arbitrary infinite subset of natural numbers. Let $N \equiv \{n_1, n_2, n_3, n_4, \dots\}$ be an infinite subset of \mathbb{N} such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Given N , we define infinite utility streams $x(N, n_4) \equiv \langle x_n \rangle$ and $y(N, n_4) \equiv \langle y_n \rangle$ as follows. For the natural numbers, $n \in \{1, \dots, n_4 - 1\}$, the utility streams contain a pair of elements (d, e) or (c, f) , and for all subsequent coordinates, the elements are either a or b . Thus the sequence $x(N, n_4) \equiv \langle x_n \rangle$ is:

$$x_n = \begin{cases} (c, f) & \text{if } n \in \{1, \dots, n_1 - 1, n_2, \dots, n_3 - 1\}, \\ (d, e) & \text{if } n \in \{n_1, \dots, n_2 - 1, n_3, \dots, n_4 - 1\}, \\ a & \text{if } n \in \{n_{2k}, \dots, n_{2k+1} - 1 \text{ for all } k > 1\}, \\ b & \text{if } n \in \{n_{2k+1}, \dots, n_{2(k+1)} - 1 \text{ for all } k > 1\}. \end{cases} \quad (5)$$

We also define the sequence $y(N, n_4) \equiv \langle y_n \rangle$ using the subset $N \setminus \{n_1\}$ in place of subset N , in identical fashion¹³,

$$y_n = \begin{cases} (c, f) & \text{if } n \in \{1, \dots, n_2 - 1, n_3, \dots, n_4 - 1\}, \\ (d, e) & \text{if } n \in \{n_2, \dots, n_3 - 1\}, \\ a & \text{if } n \in \{n_{2k+1}, \dots, n_{2(k+1)} - 1 \text{ for all } k > 1\}, \\ b & \text{if } n \in \{n_{2k}, \dots, n_{2k+1} - 1 \text{ for all } k > 1\}. \end{cases} \quad (6)$$

It is pertinent to note that once N is fixed, for any subsequence T , the sequences $x(T, n_4)$ and $y(T, n_4)$ are well-defined. We refer to first $2(n_4 - 1)$ elements of the sequences in pairs, and the remaining elements are referred to by the coordinates. An equivalent way to think of the coordinates could be as $\{-n_4 + 2, -n_4 + 3, \dots, -1, 0, 1, 2, \dots, n_4 - 1, \dots\}$.

Let \succsim be a social welfare order satisfying Monotonicity and Strong Equity. We claim that the collection of sets $\Gamma \equiv \{N \in \Omega : y(N) \succ x(N)\}$ is non-Ramsey. We need to show that for each $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and $\Omega \setminus \Gamma$. For this, it is sufficient to show that for each $T \in \Omega$, there exists $S \in \Omega(T)$ such that either $T \in \Gamma$ or $S \in \Gamma$, with the either/or being exclusive.

As the binary relation is complete, one of the following cases must arise: (a) $y(T) \succ x(T)$; (b) $x(T) \succ y(T)$; (c) $x(T) \sim y(T)$. Accordingly, we now separate our analysis into three cases.

(a) Let $y(T) \succ x(T)$; that is, $T \in \Gamma$.

We drop t_1, t_5, t_6 from T to obtain $S = \{t_2, t_3, t_4, t_7, t_8, t_9, t_{10}, \dots\}$. Hence $S \in \Omega(T)$. Observe that for all pairs, $t \in \{t_1, \dots, t_2 - 1\}$, $x_t(T) = (d, e) \succ (c, f) = y_t(S)$ by SE; for all $t \in \{t_5, \dots, t_6 - 1\}$, $x_t(S) = x_t(T) = b > a = y_t(T) = y_t(S)$. Further, $x_t(T) = y_t(S)$ and $x_t(S) = y_t(T)$ for all remaining coordinates. Then $x(T) \succ y(S)$ and $x(S) \succsim y(T)$ by SE and M. Since $y(T) \succ x(T)$, we get:

$$x(S) \succsim y(T) \succ x(T) \succ y(S)$$

Thus, $x(S) \succ y(S)$ by transitivity of \succsim , and so $S \notin \Gamma$.

(b) Let $x(T) \succ y(T)$; that is, $T \notin \Gamma$.

We drop t_1 and t_{4n}, t_{4n+1} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_6, t_7, t_{10}, t_{11}, \dots\}$. Hence $S \in \Omega(T)$. Denote the set of coordinates $\{t_{4n}, \dots, t_{4n+1} - 1 : n \in \mathbb{N}\}$ by \hat{T} . Observe that for all $t \in \hat{T}$, $y_t(S) = y_t(T) = b > a = x_t(T) = x_t(S)$; and there are infinitely many such coordinates. Further, $x_t(S) = y_t(T)$ for all remaining coordinates. Therefore, $y(T) \succsim x(S)$. Also, the first elements in the

¹³For illustration, for $N = \{1, 2, 3, 4, \dots\}$, the two utility streams are $x(N) = \{(d, e), (c, f), (d, e), a, b, \dots\}$ and $y(N) = \{(c, f), (d, e), (c, f), b, a, \dots\}$. We omit references to n_4 to simplify notation where it is obvious from the context.

$t \in \{t_1, \dots, t_2 - 1\}$ pair, $y_t(S) = c < d = x_t(T)$, implies that $x(T)$ and $y(S)$ are not comparable by M. However, (i) there are finitely many first coordinates in the pairs $\{t_1, \dots, t_2 - 1\}$ and infinitely many coordinates in \hat{T} , and (ii) for all such coordinates p in pairs, $\{t_1, \dots, t_2 - 1\}$, it is possible to pick a coordinate $q \in \hat{T}$, such that:

$$x_p(T) = d > y_p(S) = c > b = y_q(S) > a = x_q(T)$$

Therefore, $y(S) \succ x(T)$ by SE and M. Since $x(T) \succ y(T)$, we get:

$$y(S) \succ x(T) \succ y(T) \succsim x(S)$$

Thus, $y(S) \succ x(S)$ by transitivity of \succsim , and so $S \in \Gamma$.

(c) Let $x(T) \sim y(T)$; that is, $T \notin \Gamma$.

We drop t_1, t_2, t_3, t_6, t_7 and $t_{8n}, t_{8n+1}, t_{8n+2}, t_{8n+3}, t_{8n+6}, t_{8n+7}$ for all $n \in \mathbb{N}$ from T to obtain $S = \{t_4, t_5, t_{12}, t_{13}, \dots\}$. Hence $S \in \Omega(T)$. Denote the set of coordinates $\{t_6, \dots, t_7 - 1; t_{8n}, \dots, t_{8n+1} - 1; t_{8n+2}, \dots, t_{8n+3} - 1; t_{8n+6}, \dots, t_{8n+7} - 1 : n \in \mathbb{N}\}$ by \hat{T} . Observe that for the sequences $x(S)$ and $y(T)$, the pairs for all $t \in \{t_2, \dots, t_3 - 1\}$, $x_t(S) = (c, f) \prec (d, e) = y_t(T)$ by SE; and for all $t \in \hat{T}$, $y_t(S) = y_t(T) = b > a = x_t(T) = x_t(S)$. For remaining coordinates, $x_t(S) = y_t(T)$. Hence $y(T) \succ x(S)$ by SE and M. Also, observe that for comparing $x(T)$ and $y(S)$, (i) the first elements in pairs for all $t \in \{t_1, \dots, t_2 - 1\} \cup \{t_3, \dots, t_4 - 1\}$, $x_t(T) = d > c = y_t(S)$; (ii) there are finitely many coordinates, $\{t_1, \dots, t_2 - 1; t_3, \dots, t_4 - 1\}$ and infinitely many coordinates in \hat{T} , and (iii) for all first coordinates $p \in \{t_1, \dots, t_2 - 1; t_3, \dots, t_4 - 1\}$, it is possible to pick a coordinate $q \in \hat{T}$, such that:

$$x_p(T) = d > y_p(S) = c > b = y_q(S) > a = x_q(T)$$

Hence, $y(S) \succ x(T)$ by SE and M. Since $x(T) \sim y(T)$, we get:

$$y(S) \succ x(T) \sim y(T) \succ x(S)$$

Thus, $y(S) \succ x(S)$ by transitivity of \succsim , and so $S \in \Gamma$. ■

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