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AUTHOR NAMES:

Monahan, Ceire; Munakata, Mika; Vaidya, Ashwin

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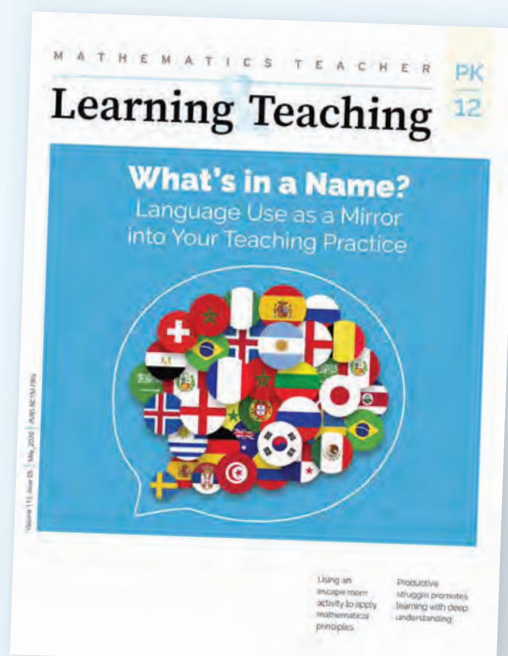
CONTACT: mtlt@nctm.org



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Engaging in Probabilistic Thinking through Play

Students explore probability in an open-ended environment by participating in two activities that employ Monte Carlo simulations.

Ceire Monahan, Mika Munakata, and Ashwin Vaidya

We present two connected activities that engage students in mathematical reasoning through creativity and play. Through these activities, we aim to encourage students to use their estimation skills while also thinking about probability and geometry through hands-on, inquiry-based explorations. Students were also asked to generate and use data to make conjectures, tying their previous knowledge about the world to the mathematics they were experiencing. The emphases on process over outcome, making connections, being inquisitive, and questioning norms are some of the characteristics of creativity highlighted in this lesson.

The term *play* is critical to our philosophy and practice. The course in which this activity was enacted is a general education undergraduate mathematics course for nonmajors and is typically taken by students

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who have little interest in mathematics or are intimidated by the subject. Therefore, our course was redesigned to be one where strict rules and formulas were de-emphasized in favor of discovery, exploration, and reasoning. All activities in this class had some aspect of creativity and play; students were not told what to do and often did not know what the exercise was leading to. It was up to students to think of and develop appropriate questions. The aim was for students to "enjoy the moment" and to have them reflect about connections to mathematical concepts.

Whereas other authors have reported on ideas for teaching probability and statistics (Gelman and Glickman 2000; Groth 2007; Jones 2005; Sharma 2015), we offer an example of a nontraditional lesson using the context of probability and geometry to reinforce students' estimation strategies and mathematical reasoning.

FRAMING THE ACTIVITIES

The two activities were part of a lesson on probability, and they engaged students in Monte Carlo methods of probability. They were implemented after a review of basic principles of probability, such as independent events, randomness, and sample size in the context of real-life examples. These activities were also set up by explorations of Fermi problems and non-Euclidean geometry. During the lesson on Fermi problems, students were asked such seemingly absurd questions as these:

- How many gallons of gasoline are used in the United States each year?
- How many revolutions will the wheel of your car make between New York City and Washington, D.C.?
- What is the area of the Pacific Ocean?

In responding to these questions, students were asked to reveal their thinking and approaches as they justified their solutions. These types of questions and

Ceire Monahan, monahanc4@montclair.edu, is a doctoral student and research assistant at Montclair State University in New Jersey. She is a former middle school teacher interested in developing innovative ways to infuse creativity in mathematics to improve student learning along with understanding best practices in preparing and supporting preservice and in-service teachers to teach mathematics.

Mika Munakata, munakatam@montclair.edu, is a professor of mathematics education in the Department of Mathematical Sciences at Montclair State University. In addition to this project on creativity in STEM, her research interests include teacher leadership, interdisciplinary STEM teaching and learning, and professional development.

Ashwin Vaidya, vaidyaa@montclair.edu, is an associate professor of mathematics and physics at Montclair State University. His research lies in the areas of complex systems, theoretical and experimental fluid mechanics, and STEM education.

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discussions about student thinking led to the activities presented here, which aim to offer other methods of estimating.

The activities were scaffolded in a way that encouraged creativity. In the first activity, activity sheets were provided primarily for record keeping (see appendix A in the supplemental materials). It was up to the students, as they took part in the activities and the whole-group discussion, to think about the meaning of the lesson and the questions they might ask. The activity sheet for the second activity (see appendix B in the supplemental materials) was developed after the first activity and incorporated some lines of questioning that students brought up in the larger group discussion.

Part I: Atlas Activity

To prepare for the Atlas activity, we first arranged eight blank posters in the form of a rectangle (four posters arranged vertically in two rows). We projected a Mercator projection of a world atlas we found online onto the eight panels of paper, with each panel representing about one-eighth of the earth. We roughly outlined and shaded the continents and other major land masses, leaving water unshaded.

In class, each group of three or four students received one of the panels. We purposefully did not reveal the representations of the panels because we wanted to encourage questioning and discovery over prescriptive lessons. Students also received diluted finger paint and sets of disposable gloves. Group members were asked to splatter paint (using their fingers) as randomly as possible for about five minutes (see figure 1) until they had a sufficient number of discernable droplets on their paper. The groups had a lot of disparity in the number and size of droplets among their panels, leading to interesting discussions later. One group had about 100 dots, and another, more than 1,000 smaller dots. We had students count the paint dots that fell on shaded versus unshaded regions and calculate the percentage of each out of total dots (see appendix A in the supplemental materials). At this stage, we hung the panels one by one in the original configuration, and only then did students see the array as representing Earth (see figure 2).

Once they saw the atlas, students considered questions that the data they collected might answer. Students saw the possibilities of using the dots to estimate the relative area of water or landmass on Earth. In the ensuing discussion, students made conjectures about how accurate this method was for estimating the percentage of landmass of Earth, what could have affected the outcome, and the considerations one must give to this experiment, given that we were representing Earth on a flat surface. Questions arose about the impact of the data obtained from the group that had more than 1,000 dots versus the one with approximately 100 dots. In fact, because each group had a different dot count, students questioned how to weigh each group's contribution to the larger data set. One panel represented only water (the Pacific Ocean). Students debated about whether to include that data set in our calculations. In the end, despite not weighing the unequal contribution of each panel, the count revealed 8,669 total dots—5,507 of which fell on unshaded areas, leading to an estimate that 63.5 percent of Earth is water. We then discussed how this estimate might be made

Fig. 1

For the first activity, group members took turns randomly splattering paint until they had a sufficient number of discernable droplets on their paper.

Fig. 2

Only when the panels were hung in the original configuration did students see that the array represented the earth.

more accurate given that Earth is not flat or how weighing the contributions of each group would affect the results. One student created a video simulation (see the video in the supplemental materials) to explore these and other percentages.

Part II: Globe Activity

In a subsequent class, students were again divided into groups of three or four for a related activity. Each group was given a "globe" beach ball (a standard-size beach ball with the globe painted on it, available online from many sources), a cup filled with finger paint, and disposable gloves. Students were instructed to go into the hallway, dip their fingers in paint, and lob the plastic globe between two people until about 100 marks were made on the globe (see figure 3). After completing the tosses, students counted the number of paint marks that fell on land and on water, recorded their results (see appendix B in the supplemental materials), and developed estimates of the percentage of Earth that is water.

During the discussion, students thought about the difference between this Globe activity and the previous Atlas activity. They compared the results on the basis of the differences in procedures and the nature of objects (poster vs. ball).

Follow-up Discussion

Our learning objectives for these activities focused on engaging students in mathematics content as they explored processes related to mathematical reasoning. We encouraged students to ask questions, pose conjectures, question norms, make connections, analyze potential sources of error, and consider other situations in which the Monte Carlo method would be a useful and sufficient way to estimate real-life phenomena. The mathematical focus was on experimental and theoretical probability as well as ratios and percentages. The discussions that followed each of these activities illuminated students' understanding of the underlying mathematics.

In the context of the probability experiment, students realized that sprinkling paint drops on the "map" was problematic and would yield inconsistent results. At this point, we discussed the difference between a uniform random distribution and the experimental approach. In future iterations of this activity, emphasis should be placed on why the procedure described here is incorrect. A follow-up activity to emphasize the difference between this haphazard sprinkling of paint and random distribution might include comparing results after projecting a clear slide with a computer-generated uniform random distribution of dots onto the maps taped to the board. Throughout the course, we emphasized the approximate nature of some of our approaches.

In an earlier lesson, we had discussed spherical geometry, so students had some knowledge of the effect of curvature on mathematical calculations. The Atlas and Globe activities provided different examples of when to consider nonlinear effects. Several students noted that conducting the experiment on a flat surface "doesn't account for the curvature of the earth" and led to an "inaccurate display of the earth" because the earth is spherical, not flat. In this context, we also briefly talked about the errors in projecting the globe onto the plane and talked about the various kinds of projections that were available, each with its own restrictions and flaws.

During these discussions, we asked students about situations in which similar estimation strategies might be used. Ideas centered on census, polls, and market research. For example, if researchers surveyed 2,000 people randomly and found that 800 used a certain brand of toothpaste, what would that mean? As another example, what considerations have to be taken into account when we hear of polling numbers? Students conjectured that we should be aware of sources of error—how many people were polled and their biases, for example.

Fig. 3

For the second activity, students tossed a beach ball globe back and forth, counted the number of paint marks from their hands, and recorded their results to develop estimates of the percentage of Earth that is water.

STUDENT REACTIONS TO THE ACTIVITIES

Student reactions to this activity were collected from activity sheets, journals, focus group interviews, and

The words our students used to convey the unexpectedness of the activity and results . . . are not . . . typically used in reaction to mathematics lessons.

notes we took during class discussions. For the class, students kept weekly journals in which they were encouraged to write about their reflections on class activities. We identified categories that emerged from this data source and present sample excerpts below.

Students' Thinking about Mathematics

Throughout these activities, we challenged students' conceptions about mathematics. In our activities, the idea of exploration was reinforced by refraining from giving students any idea about the end goal. As one student noted, "At first I didn't know what the activity was going to end with."

In fact, for our activities, students chose the questions they thought they could answer, given the data they generated. One student commented that "this type of math forces me to really think outside of the box and creatively."

Others commented on the visual representation of the activity, noting that "it keeps us engaged in the class [when] we get to visualize our assignments."

Students' Understanding about Estimation and Probability

Our mathematics content learning objective was for students to use probabilistic reasoning effectively. Students were able to connect their estimation method for our activities to other situations, such as "learning about probability and randomness and how much I personally actually use that in my day to day life."

Another noted, "I think that it represents how we can use a huge amount of data and translate it into our everyday classrooms. This opens a discussion for what else we can translate to a smaller scale and use in our own lives."

Most students seemed to understand the effect of sample size and of repeating the experiment. For others, the process piqued their interest about probability. "At first, I had trouble following; I didn't really get it. So, I did research about the Monte Carlo method. From what I understand, the Monte Carlo method is completely random. As the experiment gets more and more random, the more accurate the results [will be]."

Surprised by the Results

Our estimate using the globe was that 73.5 percent of Earth is ocean water. This was very close to the actual 71 percent. The relative accuracy of our estimate left some in awe: "I was *shocked* to see how close our results were to the actual results."

Others used similar language to convey their surprise: "I was in *disbelief* because I never would have guessed that one ocean (the Pacific Ocean, the area of which was estimated later) could count that much toward the entire earth."

Another student commented, "The earth always has fascinated me, but this answer *surprised* me more than I could have imagined. Just to think that close to half the world's surface is one entire ocean *stunned* me in more ways for me to think of."

The words our students used (e.g., *shocked*, *disbelief*, *surprised*, *stunned*) to convey the unexpectedness of the activity and results, are highlighted here because they are not words typically used in reaction to mathematics lessons. These reactions exemplify what may occur when we ask students to question norms and make connections using mathematics.

Misconceptions

Despite our intention to have the splatters of paint represent haphazard dispersion of "dots," several of our students identified the paint drops with rain drops, believing that "the point of the experiment was to show rainfall totals in the ocean versus on land." These comments were made during interviews after the whole-group discussion, so we were left wondering how we might adapt our activities given these reactions. One adaptation would be to scatter small objects, such as beans or rice grains, rather than diluted paint. Regardless, in our lesson, we would need to include a discussion about what exactly the drops of paint are being used to represent in our process of estimating. On the flip side, such misinterpretations are part and parcel of open-ended learning. We see this less as a failure and more as a potential opportunity to foster the emergence of ideas, some of which are likely to be meaningful.

CONCLUSION

The activities described in this article address many recommendations of the Common Core State Standards for Mathematics (NGA Center and CCSSO 2010), including specific content standards (6.RP.A.3, 7.PR.A.3, HSS.MD.B.5) and Standards for Mathematical Practice (SMP). By reflecting thoughtfully throughout the activities, students were able to develop mathematical habits of mind (practices), including making sense of problems and persevering in solving them (SMP 1), reasoning abstractly and quantitatively (SMP 2), modeling with mathematics (SMP 4), attending to precision (SMP 6), and looking for and making sense of structure (SMP 7). Although attending to precision was not our focus, we had a rich discussion about analyzing potential sources of imprecision.

Some of the exercises could offer a hands-on alternative to more traditional lessons in probability. Applets such as https://www.random.org/geographic-coordinates/ and http://www.geomidpoint.com/random/ provide a computer option for these types of simulations and can be performed in class or given to students as a homework assignment (although there is something to be said for doing the actual experiment).

Finally, although we chose a Mercator projection, meaningful discussions could ensue about the relative accuracy of estimates based on different projections of Earth. Examples of other projections can be found at https://www.geolounge.com/types-map-projections/.

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