7 Mapping the cultural formation of the mathematical aesthetic

In this chapter, we explore some political dimensions of the mathematical aesthetic. We draw on the work of Jacques Rancière to argue that the mathematical aesthetic must be analysed as a form of cultural politics. We claim that aesthetic practices are political practices precisely because they partake in 'the distribution of the sensible', a distribution that discloses and determines that which is held in common within a particular community (Rancière, 2004). Aesthetic practices are simply ways of 'doing and making' among many other ways of doing and making, but they are uniquely generative of forms of visibility and sensibility, and are thereby central to determining what others might call membership in a community of practice. As Rancière argues, aesthetic practices operate through a paradoxical mix of *autonomy* and *dependence*, on the one hand free from the demands of functionality and explanation (a painting is only a painting if it is *not* useful), and on the other hand entirely reliant on sensory effect (a painting is only a painting if it is perceived). We argue in this chapter that a mathematical aesthetic operates through the same paradoxical mix. Our aim ~~in this chapter~~ is to show how these aesthetic practices function in mathematics, as well as in mathematics education, and to indicate that any particular distribution of the sensible privileges certain forms of sensibilities over others. In other words, any particular drawing of the boundary between what makes mathematical sense and what does not entails a particular kind of consensus about the valuing and regulating of the senses. We will draw on our exploration of the mathematical aesthetic to argue that instead of seeking new – and possibly more invasive – forms of consensus-making in mathematics education, we might instead look for ways of perturbing current aesthetic regimes. Rancière's notions of *consensus* and *dissensus* enable us to explore current and alternate ways of making sense in mathematics classrooms. To illustrate these important notions, we begin with a

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short example from the mathematics classroom where both consensus and dissensus can be seen in action.

Classroom consensus and dissensus

Mathematics classroom discourse often aims for consensus, not simply through an emphasis on one correct answer, but also through the primacy of the alphanumeric as the ultimate form of communication. Spatial sense is typically downplayed in classroom activity, while haptic and other senses are rarely, if ever, considered. And yet mathematical consensus marshals all of the senses to make itself felt. And with equal force, the senses are the site of resistance to that same consensus. Consider, for instance, a well-known video excerpt of a grade three classroom,[1] in which a student named Sean suggests that six is both odd and even because three is an odd number and six is three groups of two. This video excerpt has been discussed and analysed at length in the mathematics education literature.[2] Ball (1993) uses it to examine the dilemma of respecting children as mathematical thinkers. Sinclair (2010) interprets the episode in terms of the contrasting intellectual passions to which Sean and his classmate Mei are committed. Here, we propose to re-read the episode with an eye to the political, that is, in terms of the acts that create and/or disrupt normative ways of making sense of mathematics in the classroom.

The class has been working with patterns involving even and odd numbers when, one day, Sean announces that he had been thinking that 'six could be both odd and even' because it is made of 'three twos'. The students discuss his proposal and dispute its legitimacy. Sean notes that not all even numbers are also odd, but that 6 and 10 are because they can be considered odd or 'unfair' groupings. He valiantly defends his assertion in the face of a growing concern on behalf of the other students. Another student – Cassandra – disagrees with Sean and goes to the board to show why, picking up the pointer and reaching up to point at the visible number line above the board. She rhythmically counts off the numbers 'even, odd, even, odd, even, odd …', as though the physical act of repeatedly banging the pointer against the number line shows why six cannot be odd. One can see in her action a rhythmic and ritual enactment of the autonomy of the even-odd number pattern. She is literally performing how the pattern has a certain automatic unfolding logic in it. One can also see in her action the way in which the body is

[1] This video is available at http://ummedia04.rs.itd.umich.edu/~dams/umgeneral/seannumbers-ofala-xy_subtitled_59110_QuickTimeLarge.mov.

[2] In earlier reports of the data, the boy was referred to as Shea.

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implicated in the performance of this autonomy. We see here an important gesture that aligns sense (as sensation) with sense (as meaning). Sean refuses to accept this binary, nor is he persuaded by the temporal enactment of odds and evens, and he returns to his partitioning-based argument.

At this point in the lesson, the teacher makes a move to reintroduce 'common sense' by asking the class to give a show of hands indicating who knows the 'working definition' of even and odd. The show of hands is another embodied gesture that makes visible the commonality of common sense, especially in this case, as it physically demonstrates a shared commitment to a definition of the mathematical concept under discussion. The teacher, after listening as Sean offers his argument another time, draws six circles on the blackboard while asking 'are you saying that all numbers are odd then?' Sean uses these circles, dividing them into three groups of two, to 'prove' to his classmates that six is also odd. The class thus moves away from the rhythmic tapping on the number line as the material terrain for establishing parity/disparity towards the discrete object-driven view of number, where each number stands on its own, individuated by property rather than by sequence. The flow of conversation is so seamless that the major ontological disruption – from a temporal, alternating definition of odd/even to a differently structured one – passes by unnoticed.

After working with other examples of even and odd numbers (10 and 21), also involving the partitioning of circles, another student (Mei), who also disagrees with Sean, sums up the concern provoked by his disruptive act, stating: '[L]ike if you keep on going like that and you say that other numbers are odd and even, maybe we'll end it up with all numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn't be even having this discussion!' Indeed, it is Mei's vision of what 'make[s] sense' that aligns with the conventional mathematical definitions of even and odd, and her eloquent argument has on more than one occasion led viewers (of the video) to comment on her mathematical sophistication.

In contrast to Mei, Sean's contribution breaks with common number sense and offers an alternative way of organising the natural numbers in terms of factors.[3] Sean justifies his suggestion that some numbers are

[3] Indeed, Sean is indirectly distinguishing even numbers that contain at least one odd factor from those that do not, the latter having the specific label of 'powers of two' in mathematics. Sean's 'even-and-odd' numbers are all the even numbers that are not powers of two. He is gesturing towards a tripartite division of numbers that consists of the totally odd, the odd-and-even, and the totally even. Such a categorisation is entirely mathematically defensible and indeed, in particular problem situations, perfectly functional.

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both odd and even by showing how these numbers are *more* than even. In other words, one might focus on how numbers like six can be generated as a set consisting of *an odd number of things*. He offers a new way of partitioning these numbers and disrupts the binary logic of even-or-odd. In pursuing this new partition of numbers, Sean troubles the current way of making sense in the classroom.

We see Sean's contribution as an act of dissensus, in that it makes visible and audible what was previously invisible and inaudible; dissensus, as Rancière (2004) proposes, 'enacts a different *sharing* of the sensible' (p. 7). Dissensus is often a short-lived moment of dispute when the distribution of the sensible is contested, when someone stands, speaks out, touches an untouchable, eats a forbidden fruit or gazes into a once-veiled object, a moment when the senses are used 'improperly' to dispute the equality of common sense. If consensus is an alignment between sense (as sensation) and sense (as meaning), then we use the term dissensus to refer to that which breaks up this alignment. Rancière (2009) suggests that a political and polemical redistribution of the senses – a 'dissensual supplement to the other forms of human gathering' (p. 32) – offers us a place to start thinking differently about community. An act of dissensus is a controversial disruption of the limits of the sensible in any given collective situation. Acts of dissensus operate on the ragged boundary between the aesthetic and the non-aesthetic; that is, they often operate as sites of non-sense, where sense is dislocated from meaning. They are border crossings, shredding the borders and divisions that currently partition the sensible. In Sean's alternative partitioning of the natural numbers, we see how dissensus is not an overturning of institutions: It does not simply reorder hierarchies of power. Dissensus is:

> [a]n activity that cuts across forms of cultural and identity belonging and hierarchies between discourses and genres, working to introduce new subjects and heterogeneous objects into the field of perception. (Rancière, 2010, p. 2)

Thus, acts of dissensus introduce new subjects and objects into the field of perception. In the previous analysis, we focused on one particular new object – a number that is both even and odd – but Sean is also a newly configured subject who is newly entangled in the concepts he perceives. The subject comes into being through both consensus (alignment with common sense) and through dissensus (divergent individuation). The senses become sites where subjects exhibiting difference and diversity are either recognized as intelligible (visible, audible, etc.) or unintelligible (invisible, inaudible, etc.).

Following this line of argument, one can unpack the term 'community of practice' in terms of *community of sense* so as to describe how cultural politics maps onto classrooms through the senses. In classrooms, the sensible is distributed and partitioned into forms that fuse visibility (or audibility, etc.) with intelligibility. By focusing on the role of the senses in delineating membership in a community of practice, we can begin to study the contingency of intelligibility to show how sense-making might be done differently. Rancière's (2009) community of sense is not about agreed-upon ways of doing things in the classroom, as in Yackel and Cobb's (1996) 'sociomathematical norms', which function 'above the senses' in that they focus almost exclusively on discourse. Rather, Rancière takes a more materialist approach by focusing on the partitioning of the sensible:

> I do not take the phrase 'community of sense' to mean a collectivity shaped by some common feeling. I understand it as a frame of visibility and intelligibility that puts things or practices together under the same meaning, which shapes thereby a certain sense of community. A community of sense is a certain cutting out of space and time that binds together practices, forms of visibility, and patterns of intelligibility. I call this cutting out and this linkage a partition of the sensible. (Rancière, 2009, p. 31)

His correlation between the senses and intelligibility offers subtle but significant insight into the way that meaning-making emerges in classrooms. The term 'intelligibility' is here used to point to the fusing of the 'true' with the 'sensible' in what is taken as *common* to the community. As we discuss later in this chapter, this correlation has influenced the kinds of mathematical practices that have become valued in policy and curriculum. We first discuss, however, the novel way in which Rancière formulates the intersections of politics and aesthetics, and then we extend this discussion to the mathematical aesthetic in order to examine how autonomy works in mathematics.

A political aesthetics

In his *Letters on the aesthetic education of mankind*, the eighteenth-century poet and philosopher Friedrich Schiller put forward a theory of aesthetics that aimed to bear 'the whole edifice of the art of the beautiful and of the still more difficult art of living' (quoted in Rancière, 2010, p. 115). It is this paradoxical coupling of art and life under the banner of aesthetics, suggests Rancière, that has made aesthetics so enigmatic and difficult to study.

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And yet it is precisely this odd coupling that has allowed the aesthetic to function so effectively as part of the political fabric of life. The aesthetic we have inherited from this tradition is radically different from other ways of embracing art, such as those that construe art as a representation or copy of the true, or those that construe art as a shaping of matter, for each of these entails an ontological divide between *the art of the beautiful* and *the art of living*. According to Rancière, the Western tradition of aesthetics confounds two oppositional concepts of sense, the first associated with the autonomy of art and the second with collective or common forms of sensibility. Thus, the aesthetic operates through the conjunction of pure 'sense' and common sense, conditioning our modes of individual perception, as well as our social institutions. In other words, political participation and 'artistic' practices are reciprocally implicated, not simply in terms of class and *judgements* of taste, but in terms of the material distribution of what is taken to be *sensible*.

> The entire question of the 'politics of aesthetics' – in other words, of the aesthetic regime of art – turns on this short conjunction. The aesthetic experience is effective inasmuch as it is the experience of that *and*. (Rancière, 2002, p. 134)

But it is not simply art qua art that partakes of the aesthetic experience, for one can study the tension of this awkward conjunction in other activities. In mathematics, for instance, one might think of this 'and' in terms of the complex ways in which form and function are mutually entailed in our sensory experiences: the way, for instance, that we attend to the formal qualities of a cube (edges, faces, angles) *and* simultaneously perceive its affordances and capacities for movement or activity (as when we work on a Rubik's cube). Or one might think of this conjunction in terms of the singularity of sensory encounters (the way sensory experiences individualize the body) *and* the making of consensus through the valuing and regulating of the senses. The eye, for instance, perceives the cube as a distinctive relationship but is simultaneously calibrated to see certain things through the forms of attending encouraged in the classroom. This powerful enfolding of politics with aesthetics operates through the senses. Decoding a cube in terms of the agreed-upon markings that designate congruence entails a falling-into-step of the senses and their alignment with common sense. Similarly, classroom tasks that direct students to correlate equations with tables of numbers determine the limits of the sensible by presenting mathematics in such forms – these tables and equations become the surface of mathematics, beyond which we cannot reach. At the same time, these tasks stage the 'proper' forms of perception that are appropriate for this community of practice.

The mathematical aesthetic, according to Rancière's use of *aesthetic*, thus acts to conjoin the purity of logical deduction with embodied sensory engagement. This can be seen beautifully in the videotapes of a mathematics lecturer discussed by Núñez (2006), in which the lecturer offers a highly formal verbal description of infinity, while at the same time performing infinity by rhythmically gesturing a repeated, linear movement away from his body. While his speech negates his own being – through the detemporalized, decontextualized and depersonalised voice of the mathematical discourse – his hand insists on carving out the iterative space. But the sensory engagement involves affect, as well. As Lockhart (2009) suggests in his popular essay, *A mathematician's lament*, a mathematical argument 'should feel like a splash of cool water, and be a beacon of light – it should refresh the spirit and illuminate the mind' (p. 68). The demand that mathematics touch our being like 'a splash of cool water', as though it were meant both to wake us up and to cleanse us, reveals how it is meant to operate through this aesthetic duality, this conjunction of enlightenment (purity) and refreshment. Rancière (2002) points to how the aesthetic regime operates in cultural formations more generally, showing how it functions as a policing force in 'totalitarian attempts at making the community into a work of art', as well as in 'the everyday aestheticized life of a liberal society and its commercial entertainment' (p. 133). Thus, aesthetics operates by folding the sensory fabric of the common, which separates inside from outside, partitioning both the social and material worlds, while entwining or coupling what is 'in' and what is 'out', so that no line of inventive flight is ever entirely free.

At work in all of this is a twisted concept of autonomy, one that grants independence to the aesthetic object and then takes it away. In other words, the duality of the aesthetic object demands, on the one hand, autonomy from functionality and explanation, while on the other hand, it emerges through a dependence on the everyday sensory modalities of embodiment. From the modernist tradition of aesthetics, we inherit an image of the art object as autonomous and its aesthetic qualities as indefinable and out of reach. Ironically, art is not a *work* of art (compare a painting by Jackson Pollock and a picture of Pollock painting in his signature style), because the aesthetic qualities are meant to be somehow contained within the art object. In other words, any labour involved in the production of art is deemed tangential to its power and presence. One can see in this approach a similarity to popular conceptions of mathematics – which we explore in more detail

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in the next section – where the mathematician is cast as someone whose insight or intuition has led to the discovery of a mathematical object, rather than someone who has laboured to produce a mathematical object.

Any discussion of the aesthetic element in mathematics has to grapple with the way aesthetic practices are conceived in relation to this concept of autonomy. An activity is considered art insofar as it partakes of this autonomy and, paradoxically, disconnects itself from its own making – 'art is art to the extent that it is something else than art' (Rancière, 2002, p. 137). For instance, Hegel argued that the artist can never entirely know the source of his or her expression, for it springs from the unconscious; similarly, in 1908 Poincaré asserted the unconscious sources of mathematical discernment. When mathematics is seen as driven primarily by aesthetic principles, as we discuss later in this chapter, the mathematician is subject to a certain *subjective anonymity* as his or her conscious presence is displaced. According to Rancière, this paradox of aesthetic sense – a lived paradox in which the autonomy and separateness of the aesthetic sense is opposed to the aspiration to *live* it as a sensibility – is actually the source of its political power. The aesthetic operates through the dream of an unavailable ideal form that must be made flesh and possessed as reality. And although we are concerned here with how this maps onto experiences of mathematics teaching and learning, it is important first to understand the ways in which this complex tradition of aesthetics has come to figure more generally in our everyday lives. One could argue, for instance, that affluent segments of Western society now look at the world and their community as though they were forms of art. Through ornament, industrial design and fashion, life – for some – has become increasingly aestheticized. To live life 'properly' in contemporary Western society is to appreciate the aesthetic qualities of one's environment and to produce or communicate in ways that reveal one's capacity for 'free' expression. Aesthetics removes art from the world through its autonomy but then inserts it back into the material world and demands that we live this autonomy through acts of freedom. The aesthetic regime of art shuttles back and forth between two scenarios in which art and non-art are linked – each with its own vanishing point or point of collapse: in one, life consumes art, and in the other, art denies life.

The paradox of aesthetics seems to thrive in contemporary consumer culture. We continue to invest in modernism's partitioning of the perceptible, whereby the commodity becomes fetishized by means of its being perceived as an aesthetic object. Indeed, an emphasis on design, as opposed to consumption, fosters a social commitment to art as a mode of collective education, whereby the population may be educated – through

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the senses – in how to live an aesthetic life. This occurs in the everyday consumer culture by which we obtain sustenance, and also in how we collect what were once prosaic objects and value them as antiques. Education figures prominently in the operation of this aesthetic regime. Education is devoted to enlisting students into particular habits of sensing and to ensuring that they internalize these forms of common sense. We argue in the following section that the mathematical aesthetic must also be read in terms of this political plot, showing how the aesthetic regime of mathematics operates through this duality by first claiming that *mathematics* partakes of the autonomy of the aesthetic and then insisting that one must live this aesthetic as a form of life.

The mathematician's sensibility

Instead of being concerned with actual senses (or a lack thereof), the literature on aesthetics and mathematics reverts to a mystification of the mathematician's sensibilities, as though the mathematician possessed some sort of intrinsic capacity to see or perceive (metaphorically, of course) either the immaterial entities of an ideal mathematical world or the masked mathematical structure of the physical world. According to this image, the mathematician is able to reconfigure the fabric of sensory life because he or she *perceives* what was previously invisible or insensible,[4] whether that be the rules governing some sort of physical process or the behaviour of a sequence of numbers. Rather than simply pointing out how this literature is a self-legitimizing discourse, our approach aims to unravel the specific means by which the aesthetic regime of mathematics ensures the visibility of mathematical objects and makes them available to thought. Lockhart's (2009) description of mathematics as an art form, and in particular the idea of inserting a line into a diagram, captures this common discourse in the literature concerning mystery and visibility:

> Now where did this idea of mine come from? How did I know to draw that line? How does a painter know where to put his brush? Inspiration, experience, trial and error, dumb luck. That's the art of it, creating these

[4] As we discuss in Chapter 8, the metaphorical aspect of this mathematical perception reached a high point in the eighteenth century, when it was considered by some to be advantageous for a mathematician to be blind. The blind mathematician would have unfettered access to the ideal, abstract objects of mathematics, including geometrical ones. These philosophers also argued that a lack of sight might compromise the mathematician's openness to the world around them, thus diminishing their moral sensitivity. The interplay between sense and common sense here achieves a staggering and polarised simplicity.

> beautiful little poems of thought, these sonnets of pure reason. There is something so wonderfully transformational about this art form. The relationship between the triangle and the rectangle was a mystery, and then that one little line made it obvious. I couldn't see, and then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing, and change myself in the process. Isn't that what art is all about? (p. 27)

The notion of pattern perception has emerged as a dominant mode of sensing in mathematics. For example, in his attempt to define mathematics as the 'classification and study of all possible patterns' (p. 12), the mathematician W. W. Sawyer (1955) implies that the heuristic value of mathematical beauty stems from mathematicians' sensitivity to pattern and originates in their belief that '*where there is pattern there is significance*' (p. 36; *emphasis in original*). Sawyer goes on to explain the heuristic value of attending to pattern:

> If in a mathematical work of any kind we find that a certain striking pattern recurs, it is always suggested that we should investigate why it occurs. It is bound to have some meaning, which we can grasp as an idea rather than as a collection of symbols. (p. 36)

Sawyer's claims resonate with how Poincaré describes the mathematician's special aesthetic sensibility as a sensibility towards pattern, which is viewed broadly as any regularity that can be recognized by the mind. For Poincaré, the mathematician is not only able to recognize regularities and symmetries, but is also attuned to look for and respond to them with further investigation. In an essay published late in his career, Alfred North Whitehead revisits Plato's ethical, or perhaps political, disquisition on mathematics and suggests an even grander visibility offered by pattern. Whitehead seems bewitched by the axiological connection Plato made between mathematics and the good and, having admitted the incompleteness of Plato's argument, offers his own:

> We cannot understand the flux which constitutes our human experience unless we realize that it is raised above the futility of infinitude by various successive types of modes of emphasis which generate the active energy of a finite assemblage. The superstitious awe of infinitude has been the bane of philosophy. The infinite has no properties. All value has the gift of finitude which is the necessary condition for activity. Also activity means the origination of patterns of assemblage, and mathematics is the study of pattern. Here we find the essential clue which relates mathematics to the study of the good, and the study of the bad. (1951, p. 674)

For Whitehead, thinking about pattern is, at its core, an ethical activity, insofar as it concerns our experience, which is shot through with the insistence of finitude. Whitehead is much less concerned with promoting scientific advances (to which the utility of mathematics most often contributes) than he is with promoting creative, flexible and non-dualistic thinking about the 'human condition'. While Whitehead saw an affinity between mathematics and moral development, or at least what he took to be part of the 'human condition', most people who consider mathematics and morality in the same breath take a more cautious point of view (Hersh, 1997).

This emphasis on pattern, symmetry and regularity underscores the kind of autonomy that epitomises many accounts of the mathematical aesthetic. Mathematical patterns can be characterized as automata, in that they operate according to an intrinsic logic or rule, independent of outside stimulus or human intervention. No matter how much material force one can muster, one cannot disrupt or alter the unfolding of the pattern. A mathematician might 'explain' a pattern with reference to the actions or operations that might be used to produce it (adding, multiplying, tripling, etc.), but this activity or labour does not engender the pattern. Automata are 'self-acting', and patterns are self-engendering. The emphasis on *detecting* patterns demands that the mathematician perceive or sense that which is independent of his or her labour. The mathematician must 'grasp as an idea' that which is autonomous – that is, the mathematician must internalize the autonomy and *live* the mathematical aesthetic as a form of life. According to Rancière, this is precisely how an aesthetic regime operates – by insisting on the conjunction of two oppositional concepts of sense, the first associated with the autonomy of expression (or, in this case, pattern) and the second with the enactment of a common form of sensibility (in this case, the mathematician's). This impossible demand to *live* that which is autonomous is reflected in the contemporary literature on the mathematical aesthetic. As Lockhart (2009) suggests:

To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion – not because it makes no sense to you, but because you *gave* it sense and you still don't understand what your creation is up to; to have a breakthrough idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be *alive*, damn it. (pp. 37–38, *italics in original*).

Poincaré (1908/1956), however, suggests that only the most creative mathematicians have access to an aesthetic sensibility. Similarly, Hardy (1940) claimed that 'the aesthetic appeal of mathematics may be very real for a chosen few' (p. 86). Any attempt to rethink the mathematical aesthetic in materialist terms has to reckon with this common inclination to assign a *selective* sensibility to the mathematician. Russell's (1919) famous quotation is frequently cited as an example of how this sensibility is expressed: 'Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture' (p. 60). A cold and austere beauty is one that is pure and independent, or erases all traces, of the hands that made it. Moreover, the stone cold sculpture shows no interiority and no emotion, and the matter is hard and resistant to the form it is given. And yet, like the quotation from Russell, a focus on mathematicians' aesthetic considerations is frequently offered as a counter to the usual emphasis on purely epistemological concerns about truth and certainty. The historian of mathematics Morris Kline (1953), for instance, points out that aesthetic concerns not only guide the direction of an investigation, but also motivate the search for new proofs of theorems that were already correctly established but lacking in aesthetic appeal. Kline concludes that this aesthetic motivation is a definitive sign of the artistic nature of mathematics. The distinction between functionality and autonomy operates in this judgement, whereby a mathematical proof becomes aesthetic as it is granted a certain autonomy, that is, as it comes to be of no apparent everyday value. In other words, once we have a proof, what is the *use* of another? In not being useful, it is deemed aesthetic. Lockhart's (2009) lament for the loss of mathematics-as-art reveals a similar distinction, as he distinguishes the 'mundane "useful" aspects [that] would follow naturally as a trivial by-product' from the more central aesthetic activity of real mathematics. The latter is precisely how the mathematician Wolfgang Krull (1930/1987) characterizes mathematics, by contrasting epistemic concerns about truth and logical consistency with aesthetic engagement:

> Mathematicians are not concerned merely with finding and proving theorems, they also want to arrange and assemble the theorems so that they appear not only correct but evident and compelling. Such a goal, I feel, is aesthetic rather than epistemological. (p. 49)

The epistemic in terms of the determination of true or false is here aligned with the functionality of mathematics, whereas being 'evident and compelling' is aligned with the aesthetic. This distinction construes mathematics as both autonomous (evident) and affective (compelling) in its aesthetic dimension. Indeed, we are compelled to submit to mathematics only when it achieves this aesthetic dimension, for it is only then that it truly embraces its autonomy. Rota (1997) draws attention to the way in which aesthetic descriptors used by mathematicians may, in fact, represent veiled ways of

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sustaining an image of mathematics as immaterial and 'immune' to the sensual:

> Mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgement of the fuzziness of this phenomenon. […] This cop-out is one step in a cherished activity of mathematicians, that of building a perfect world immune to the messiness of the ordinary world, a world where what we think *should* be true turns out to *be* true, a world that is free from the disappointments, the ambiguities, the failures of that other world in which we live. (pp. 132–133; *emphasis in original*)

This tension between the mathematical aesthetic and the messiness of the ordinary world reflects Rancière's depiction of an aesthetics that partitions the sensible into ~~access and denial of access~~. According to Rota, mathematical beauty is a way of sustaining an untouchable or unreachable sensory realm, for it is through this aesthetic judgement that the ideal becomes real. The aesthetic grants mathematics a sensory aspect, while simultaneously denying access to this encounter for all but a few.

In 1940, G. H. Hardy published what arguably became the most widely read inquiry into the mathematical aesthetic. Hardy was primarily interested in defining mathematical beauty as it exists in mathematical products, particularly in proofs. He proposed a somewhat complex scheme that distinguished 'trivial' beauty – which can be found, for instance, in chess – from 'important' beauty – which can only be found in serious mathematics. Serious mathematics requires depth and generality (scope and reach) if it is to be significant, but because none of these aesthetic qualities can be defined, only those with a 'high degree of mathematical sophistication' (p. 103) can recognize them. Such mathematicians will find a mathematical idea significant when it can be 'connected, in a natural and illuminating way, with a large complex of other mathematical ideas' (p. 89). It is the clause 'natural and illuminating' that inscribes a common ('natural') form on a particular way of sensing and/or making intelligible ('illuminating').

Hardy illustrates his notion of mathematical beauty with two examples: Euclid's proof of the infinity of primes and the Pythagorean proof of the irrationality of $\sqrt{2}$. These two proofs appear frequently in the literature as particularly fine examples of beautiful proofs. Dreyfus and Eisenberg (1986) showed five different proofs of the claim that $\sqrt{2}$ is not rational (all using indirect reasoning) to a group of mathematicians who were asked to identify the proofs that were most elegant. All of them selected the same pair and justified their choice in terms of the perceived simplicity and minimal amount of background knowledge required to understand the proof. Both of these judgements – simplicity and minimal assumed knowledge – point to particular desires that fuel the mathematical aesthetic. On the one hand, simplicity is contrasted with unnecessary adornment and a preference for truth to be present and singular, without difference or complication. On the other hand, minimal assumed knowledge relates to the desire that mathematics ultimately concern pure reason rather than knowledge, because knowledge is always tainted with the particularities of its historical context. The tension revealed within these desires echoes the larger tension of aesthetics more generally. But what remains unexamined in the Dreyfus and Eisenberg study is the way that simplicity and minimal assumed knowledge are valued ~~*only* in relation to the powerful result that emerges through these two famous proofs~~.

In other words, we need to explore further what is actually entailed in these two proofs. They are both proofs that deploy contradiction or indirect reasoning, each beginning with the positing of the opposite of that which one aims to prove. Although not all such proofs are deemed beautiful, the act of beginning with the opposite claim is a highly aesthetic move, in that doing so enacts a kind of feigned indifference or autonomy with respect to the truth of the claim. It is as though the speaker refuses to push hard for the claim, generously offering to indulge the other side ('fine, we'll set my own claim aside, and we'll go along with yours …'), much like Socrates might have done. This discursive move immediately performs a kind of autonomy by setting up a distance between the aim of the proof and the manner or direction of it. This distance ~~identifies~~ the proof ~~as~~ an appearance or performance, and we are suddenly invited to perceive the proof as an aesthetic object. Perhaps the conclusion of the proof – that being the negation of the original claim or assumption – wraps the end onto the beginning in such a way that closure and smoothness become *felt* aspects of the proof. The proof becomes a surface folding back on itself – a perfect, opaque ball – ~~which is~~ to be cherished and handled like an art object.

However, perhaps these two proofs are beautiful because they each produce the unexpected – that being the object that was previously denied existence. In the case of the infinity of primes, yet another larger number is created, literally cobbled together from a collection of commonly held primes, and this number must either be prime or be divisible by a prime number larger than the ones originally posited as the complete, finite set. We hear the speaker say, 'Do you see it now? You said it didn't exist, but I have shown you what it would have to look like.' Another larger prime

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can always be generated from any finite set of primes; the 'beauty' lies in how the new prime comes forth autonomously through logic, rather than through the labour of its material construction. The proof itself functions in redistributing the sensible and partitioning what is taken to be real. In the second case, a similar creation – that being the irrational number – plays havoc with the ontology of number and emerges, as though by magic, as a new being. The incommensurable – that is, the 'properly' imperceptible – is shown to defy our common sense, and we must recognize that there is a new way of delineating between the sensible (rational) and what was previously taken to be the non-sensible (irrational). We can see that these two proofs deal closely with the partitioning of the senses and the redistributing of what is taken to be 'common' sense.

Netz (2009) makes evident the ways that communities of sense-making in mathematics have been radically different over time, arguing that written mathematics in the time of Archimedes (from about 250 BC to 150 BC) had a distinct style that differed markedly from both that of other ancient Greek periods (including that of Euclid), as well as that of contemporary writing. Before proceeding, seeing as we are going to compare the Alexandrian style with the contemporary one, it is important to point out that Archimedes communicated his mathematics through personal letters and not through journal articles. One may argue that the mode of communication marks the essential difference between the two styles we want to compare. Nowadays, mathematicians are also permitted to communicate through letters (or emails), and their style of writing in these cases differs drastically from that of their more formal writing. However, we think it is still worth comparing the two styles, in part because the Alexandrian letters were the 'common' form of preserving mathematical knowledge, and they communicate complete results through theorems and proofs.

The aesthetic elements that Netz proposes for the Archimedean style are as follows: narrative surprise, mosaic structure, generic experiment and a certain 'carnivalesque' style. Netz borrows the colourful adjective 'carnivalesque' from Bakhtin, for whom it describes a literary mode in which humour and chaos are used to subvert and liberate assumptions associated with a dominant style. These elements are manifest in Netz's reading of Archimedes' account of spiral lines, which is devoted to the proof that the area under the segment of a spiral equals one-third the area of the corresponding circular sector. Although Hardy might have been tempted to judge this proof in terms of its being natural and illuminating, Netz is less concerned with evaluating the result in terms of its aesthetic qualities than he is in analyzing the particular style with which Archimedes relays the result.

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Netz argues that narrative surprise can be seen in the very introduction to the problem, which arises abruptly when Archimedes introduces the spiral as 'a special kind of problem, having nothing in common with those mentioned above' (p. 3). Netz shows that this kind of abrupt transition is characteristic of the Archimedean style and reoccurs throughout the letter, even as the proofs are given. It is not until proposition 24 that, as Netz writes, 'the treatise as a whole makes sense', and the enunciation of the result is given 'in economic, crystal-clear terms – the first simple, non-mystifying enunciation we have had for a long while' (p. 10).

Surprise operates here as though we were following an unfolding narrative, and the readers of the narrative are meant to be engaged precisely because they do not know what will happen next. Surprise, in this sense, must, by definition, be completely at odds with a text that unfolds according to a series of deductive implications, because these must 'follow' by necessity. From a pedagogical perspective, promoting surprise may seem counter-productive, given that we surely want the students to understand the ratiocination of the proof rather than submit to the intrigue that compels one to turn the pages of a detective novel. And yet, surprise often accompanies important learning experiences (Movshovitz-Hadar, 1988). Nunokawa (2001), for instance, argues that surprise is a critical factor in good mathematics instruction and that one could plan surprises in lessons by attending to the gaps between conjectures and realizations.

Dreyfus and Eisenberg (1986) claim surprise as one of the important aesthetic qualities of a mathematical problem. Their list includes: 'its level of prerequisite knowledge, its clarity, its simplicity, its length, its conciseness, its structure, its power, its cleverness, and whether it contains elements of surprise' (p. 3). And yet, surprise cannot be scripted or anticipated. All the characteristics of the mathematical aesthetic – clarity, brevity, elegance, conciseness – lack significant impact if a feeling of surprise is not also engendered:

> The conclusion of such a powerful argument tends to contain an element of surprise for anyone who did not know the argument before. This surprise, in turn, is a further contributor to the aesthetic appreciation of the argument; mathematicians, similar to the spectators of a magician, like the unexpected, at least as long as they consider they have a fair chance at understanding the reasons behind the surprising conclusion. The factors contributing to the aesthetic appeal of a solution or proof are thus connected to each other; they almost follow naturally from each other: clarity – simplicity – brevity – conciseness – structure – power – cleverness – surprise. (p. 6)

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Stanley (2002), however, has argued that surprise has to be seen as 'an event of emergence' and that those who are surprised must be 'prepared to be surprised', in that surprise occurs only when there is a discrepancy between expectations and experiences (p. 15). The word *surprise* has French and Latin roots in *surprendre* (to over-take) and *prehendere* (to grasp or take, as in *prehensile*), respectively. The word surprise came to refer to 'a feeling caused by something unexpected' in the sixteen century, thus combining the affective with the epistemic. As Stanley suggests, surprise is a deeply relational event, emergent through the interaction of different bodies: '[S]urprises are event-full moments or happenings' (p. 15). In particular, surprises operate through blind spots and the perturbing or subverting of other limitations to the senses.

We find that surprise is thus an event of dissensus, in that the delineation of the sensible – that which is visible, audible, intelligible – is altered and redefined. The logic of consensus is undone when that which was taken to be invisible or inaudible is made visible or audible by an act of dissent that enacts a different kind of sharing of the sensible. Political disruptions of the sensory self-evidence of the 'natural' order of life will always entail an aesthetic component of dissensus, just as art that breaks with the limits of speech and perception will reconfigure the space of political participation. If consensus is an alignment between sense (as sensation) and sense (as meaning), then we use the term dissensus to refer to that which breaks up this alignment. One can see how surprise relates directly to a theory of the body in mathematics and underscores the power of dissensus to motivate the kinds of judgements we have found in the literature on the mathematical aesthetic. We are arguing here that surprise should be dislocated from the individual and seen as an event or happening that recombines heterogeneous materialities and redefines the contours of the sensible. These popular mathematical proofs do not simply allow an individual an expanded capacity to sense – for instance, in being suddenly able to touch something that one could not touch previously. During an experience of surprise, an individual assemblage is literally 'over-taken' by new material assemblages. A surprise is an event through which two or more bodies interpenetrate in new ways, and a new assemblage emerges. Bodies mix and intermingle during surprises in ways that bring forth immanent tensions and new surface effects. Thus, surprise is a crucial facet of the mathematical aesthetic, because it operates through sense (as meaning) and sense (as corporeal activity). But surprise is also a form of dissensus, because it disrupts that which is expected. Surprise is a place or site of breakthrough, making it both what produces new bodies and also what must be presupposed for

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corporeal activity. Much of the literature on surprise, however, tends to package it as an aesthetic *judgement*, rather than studying the corporeal activity involved in it.

Netz explains that surprise operates in the Archimedean text because there is no obvious thread through the proofs that are offered, and the spiral is not even defined until halfway through the letter. Before getting there, we have a 'surprising sequence going from physics through abstract, general geometrical observations, via the geometry of circles and tangents, and finally, leading on to a *sui generis* study of arithmo-geometry, none of these being relevant to any of the others' (p. 9). Netz sees the extensive use of calculations and physics (Archimedes' spiral requires the motion of two lines for it to be called into being) as a breaking of genre boundaries and the ungoverned sequence of seemingly unrelated material as leading to a style of surprise and mosaic structure that contrasts greatly with the linear, axiomatic presentation found in contemporary mathematics. In addition, in contemporary mathematics, efforts are made to signpost the general structure of the argument, so that the reader knows how different tools – and, especially, different lemmas – are being used. This pedagogical style seems to be completely absent in the Archimedean letter.

Netz claims that Archimedes intentionally chose an obscure and 'jumpy' presentation so as to 'inspire a reader with the shocking delight of discovery, in proposition 24, how things fit together; so as to have them stumble, with a gasp, into the final, very rich results of proposition 27' (p. 14). The Archimedean writing style might thus be described less as being in pursuit of the true or the good, and more as being designed to produce a highly satisfying emotional reaction, much in the same way we expect a good detective novel to work. Furthermore, Netz points to the way in which we can attend to this Archimedean treatise in terms of the novel and somewhat exotic focus on the spiral, which he was the first to study, and which involves boundary crossings not customary in Euclidean geometry, where time and motion are customarily strictly forbidden. By means of these stylistic elements – which are evident in the extensive number of examples of mathematical writing by Archimedes and his contemporaries that Netz provides – a mathematical style emerges that contrasts markedly with the contemporary one. We have already hinted at some of the differences, but it would be misleading to neglect one difference upon which Netz elaborates at length, namely, the way in which Archimedes' mathematical writing style was influenced by, and in turn influenced, the Hellenistic literary style in poetry. It would be difficult to make a similar kind of argument today (unless one wants to consider the works of groups

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such as Oulipo), but in articulating the central tensions of both literary and mathematical cultures, Netz provides insight into the way in which a different style might have been possible in the past – and, indeed, might be possible again in the future.

In this section, we have explored contemporary mathematical sensibility and the particular forms of consensus that have been articulated by mathematicians. We used Netz's exploration of Archimedean mathematics as a point of comparison that exemplifies a radically different mathematics community of sense – and, in doing so, we have tried to show not only how the current community of sense is a choice (and not a necessary consequence) of the discipline, but we also want to ask what acts of dissensus might look like today. Such acts occur quite regularly in mathematics, in fact, and can be seen perhaps most clearly any time decisions are made about what counts as mathematics or how mathematics is different from other disciplines. One rather highly public example revolved around the Jaffe and Quinn debate (1993) that concerned boundary-making between the disciplines of mathematics and mathematical physics (a debate that would have been of particular interest to Archimedes!). At stake for Jaffe and Quinn was the safeguarding of mathematical rigour against speculation as it occurs in current interactions between physics and mathematics. Their paper provoked much debate, resulting in 16 responses by leading mathematicians in the subsequent ~~volume~~ of the *Bulletin of the American Mathematical Society*. While the issues raised were varied, much of the discussion turned on the separability of conjecture and proof. Common sense clearly distinguishes the two – despite Imre Lakatos's insistence on their fundamental dialectic nature – with proof being the only currency of truth. If conjectures can be published, as well as credited, as they have been by some theoretical physicists, then a new currency will circulate. On a more subtle level, if conjectures can be published, as well as credited, then the contingency and plasticity of formal mathematics will become far too visible.

In the next section, we continue our exploration of the theme of what counts as mathematics and mathematical activity in the context of school mathematics and, in particular, with respect to policy and curricular documents. We will show how the new consensus concerns the politics of becoming in the mathematics classroom.

New standards of curricular consensus

The struggles that have played out in mathematics curriculum reform over the past 20 years are frequently described as being highly political, in that

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what is at stake are the competing values of the traditional and reform camps. In his analysis of mathematics curriculum, Popkewitz (1998, 2004) focuses not on the politics of math education as it materialized in the so-called math wars, but on the often-neglected politics of pedagogy where the standards of reform are enacted and, as he argues, also produced. For those who have been involved in arguing for alternatives to traditional pedagogies that have disenfranchised large groups of learners and teachers, Popkewitz's analysis may seem counter-productive. Yet, his critical perspective ~~gets around~~ the ~~rhetorical~~ claims that one practice is better than another, because no pedagogy can be entirely just and equitable. Popkewitz helps us reflect on how the reform movements of the last few decades entail consequences that 'divide, demarcate, and exclude particular children from participation' (p. 1). More specifically, he considers particular reform practices that may actually reduce the 'range of phenomena for scrutiny, action, and critical thought' (p. 18).

Popkewitz organises a significant part of his critique around the term 'alchemy', which he uses to describe the way in which school subjects are formed through a transmutation of academic knowledge, where the governing principle is no longer, say, mathematical knowledge, but now mathematics pedagogy. Brown and McNamara (2011) provide a stunning example of this transmutation in their description of the changing identities of primary school teachers during the years in which the U.K.'s National Numeracy Strategy was being implemented. All research participants were asked the question 'what is mathematics?' in each of their four years of training. Their answers at the outset of their studies were clipped and numerically oriented, but later they broadened to responses such as 'exploring number, exploring shape' and 'comparing multiple solution strategies'. By the end of their training, their statements revealed a conception of mathematics that primarily involved good management of activity and commodified curricular performance standards. The authors make the insightful observation that pedagogical forms (the use of manipulatives or line graphs or drill sheets) came to stand in for the mathematics itself: 'The presentation of the activity seems to provide a way of locating mathematics, yet the activity seems to be clouding the teacher from alternatives. The pedagogical form becomes the mathematics itself such that it is otherwise "impossible to teach"' (p. 113).

Popkewitz (2004) uses the term 'inscription devices' to refer to the kinds of pedagogical forms that Brown and McNamara name in their study. For Popkewitz, alchemy happens through an 'assemblage of inscription devices that translate and order school subjects' (p. 2). The significance

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of inscription devices is that they make visible the thoughts of a child in ways that make them amenable to governing. As an inscription device that emerged in the 1980s to counteract excessive attention to procedural thinking and memorization of facts and to reflect the nature of the mathematical discipline better, 'problem solving' also changes the features of a child's inner characteristics and capacities that are deemed salient, and it functions to demarcate, preserve and make administrable these features. One can argue – and people have argued – whether or not the particular inscription devices of reform mathematics are 'better', but instead Popkewitz draws attention to the inevitable pedagogizing of mathematics that has ensued. Indeed, in the U.K. context, the alchemy made it easier – in the eyes of the teachers – to teach 'mathematics' (which had become drill sheets), but the opposite seems to hold true in the U.S. context, where reform practices seem to make 'mathematics' (which has now become problem-solving and group discussions) much more difficult to teach. But in both cases, one can trace the way that the mathematics becomes a set of 'commodities exchanged in the educational marketplace' (Brown & McNamara, 2011, p. 126), which are directly linked to particular forms of social regulation.

In the U.S. context, the particular alchemy on which Popkewitz chooses to focus seems especially complex, in that at least some of the educators involved in promoting inscription devices such as problem solving and communication drew their pedagogical forms directly from the philosophy of mathematics. More specifically, it was Lakatos's *Proofs and refutations*, in which an extended instance of historical mathematical practice was compellingly described through an imaginary dialogue theatrically set in a mathematics classroom, that inspired many of the inscription devices. Lakatos criticises the deductivist approach encoded in the formalist philosophy of mathematics, in which mathematics 'is presented as an ever-increasing set of eternal, immutable truths' (p. 142). In contrast, he offers a more fallibilist approach, and he characterizes this approach by describing the methodology of proofs and refutation, which is a general heuristic pattern of mathematical creation that consists of several stages from primitive conjecture to 'proof' to the consideration of counter-examples that result in an improved proof.

The imaginary dialogue presented in the Lakatos book consists of a historically inspired account of the Euler-Descartes formula for polyhedra. The whole method of proofs and refutations centrally involves the creation of putative counter-examples, which have become known as 'monsters' (polyhedra that do not fit the formula relating the number of vertices, edges and faces – V-E+F=2 – such as the cylinder), and the barring of these monsters (which occurs through changes in the definition of terms such as polyhedra). The simple, familiar cylinder now becomes a monster in the face of the desired formula. Does one abandon that beloved relation or make a new partition of the sensible in which the cylinder is no longer a polyhedron? Indeed, the nineteenth-century mathematicians involved in this work decried the twisted, nonsensical monsters proposed by their colleagues, begging for a return to the tamer polyhedra considered (and possibly intended) by Descartes and Euler. Monster-making and monster-barring compromise the autonomy of the polyhedron concept. Finding these monsters, that is, and producing these counter-examples can be seen as acts of dissensus. The sensory disorientation produced by such acts can be deeply disturbing, as Lakatos exemplifies when Delta, one of the characters in his play, recoils in horror from Alpha's 'monster' of nested cubes: 'I turn in disgust from your lamentable "polyhedra", for which Euler's beautiful theorem doesn't hold. I look for order and harmony in mathematics, but you only propagate anarchy and chaos' (p. 21).[5] But once the process of proofs and refutations produces its provisional partitioning, the polyhedron must emerge, autonomous once more, delivering itself back to the real world of tangible objects. Eventually, at least for now, the sensible nature of the three-dimensional objects under consideration (often presented in two-dimensional perspective on paper) succumbs to the imposing austerity of the formula, which itself eventually pursues its own line of flight when applied to topological spaces.

As Pimm, Beisiegel and Meglis (2008) point out, if Lakatos's main argument was that 'progress at the frontiers of mathematics *does not occur* by a deductive process but rather, by the very heuristic process' exemplified in his dialogue (p. 474, *emphasis in original*), there is still a long way to go before anything can be claimed for the *learning* of mathematics. Further, not only is the Socratic-like dialogue not an accurate representation of discourse, it is also not, by any stretch, meant to be an accurate representation of a mathematical classroom. That being said, many mathematics educators have jumped to facile analogies between a long, drawn-out, historical mathematical production involving expert mathematicians producing new ideas and a single classroom lesson involving children learning known ideas. For example, Ernest (1991) argued that the teacher and students should engage in ways identical to those in Lakatos's dialogue, specifically posing and solving problems, articulating and confronting

[5] Lakatos is paraphrasing Hermite, who was writing a letter Stieltjes 'with a shudder of disgust' about the 'plague of functions' that the latter was offering as counter-examples of functions that are continuous but have no derivatives.

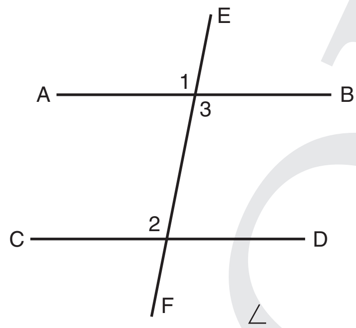
assumptions, and participating in genuine discussion (p. 208). Lampert (1990) applied the dialogue to school mathematics in her experiment to test whether the qualities of Lakatos's historical mathematics account could be observed in a classroom setting. The experiment was a success, in that the students 'learned to do mathematics together in a way that is consonant with Lakatos's and Polya's assertions about what doing and knowing mathematics entails' (p. 33). Lampert's work was later used to model and justify – using a Lakatosian dialogue – the NCTM Standards' vision of a mathematics classroom (Yackel & Hanna, 2003).

But the 'doing and knowing' of mathematics was based almost entirely on the *form* of Lakatos's assertions, namely the dialogue between students and teacher. The *content*, which involves the formulating of definitions, the creating of lemmas, the stretching of concepts, the barring of monsters, the bickering between 'students', the questioning of taste, the political and intellectual accusations, and the historical links, were taken to be epiphenomenal. Thus, the policy initiatives took up the book in a way that chose certain forms of participation to be pedagogically valuable. The central disciplinary component of Lakatos's work was left behind in favour of the new inscription device of classroom discussion.

Of course, political actions do not just occur on the level of large-scale policy issues, like 'problem solving'. A much narrower, and perhaps more mundane-looking example, can be found in the introduction of the two-column proof in American high school geometry courses in the early twentieth century. Herbst (2002) shows how this inscription device 'helped stabilize the geometry curriculum by melding together the proofs given by the text and the proofs expected from the teacher' (p. 285). Like 'problem solving', the two-column proof determines the material practices that are deemed salient, while also distinguishing common sense from non-sense and, in doing so, classifies children according to their capacity to perform accordingly. Like 'mathematics for all', the two-column proof aimed to meet the demand that 'every student should be able to do proofs' (p. 285). The two-column proof invokes a very specific material practice, in which arguments are made in rows and columns instead of in a more narrative style. All statements must fit in some cell of the table, and no statement that is unrelated to the properties of that cell can be written. Furthermore, *every* statement has to have a reason. In fact, this format involves a necessary detachment from engaging materially with the diagram, in that the statements supposedly mediate one's interaction with the diagram, because one engages with the notation, or labelling, and the utterances about these labels, rather than the physical markings of the diagram, and in doing so, Propostiton XIX. Theorem

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal*

(Converse of Prop. XIV.)



Given parallel lines *AB* and *CD* and the cor. ∠s 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
$\angle 1 = \angle 3$.	Vertical ∠s are equal..
$\angle 2 = \angle 3$.	Alt. int. ∠s of ∥ lines are equal.
$\therefore \angle 1 = \angle 2$	Things equal to the same thing are equal to each other.

Q.E.D.

Figure 7.1. A two-column proof on corresponding angles. (Photographed by the author from Schulze and Sevenoak, 1913, p. 53.)

the format keeps hands away from the image. It also creates a strict visual divide (the left and right columns) between utterances and justifications, thus performing an extraction that forces the students to divorce reasoning from expression. The impact on the hand and the eyes is immense.

Its material consequences extend to the concept of proof, so that every statement has – and has to have – a reason and vice versa; perhaps most stunningly, the means of discovery appear nowhere and, as such, remain completely separate from the logic of justification. The chain of events is illuminating: Educators decide that it is important for students to write their own ‘original’ proofs, rather than memorize and copy Euclid’s, but then they realize that this is quite difficult, so they find a way of making proof-writing more accessible through the two-column inscription device. The result is that students learn to write two-column proofs in which the *what* and the *why* have been spatially separated. The question Popkewitz

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invites us to ask is how such alchemy changes the way a child's inner characteristics and capacities are viewed. In the regime of the two-column proof, the divisions between those who can prove and those who cannot is altered radically. If earlier a student could *not* prove because proving was a challenging task, now the blame shifts to the student, who cannot prove because of an inability to follow simple, logical steps. As soon as the problem shifts to the individual, it becomes possible to objectify the learner: 'The mapping of children's activities, such as problem solving, simultaneously creates a mapping of the individual who does not 'fit' or act as a problem solver and is inscribed as *the child left behind*; (Popkewitz, 2004, p. 5, *emphasis in original*).

A decade following the NCTM Standards' focus on problem solving, in 2001, *Adding it Up* (Kilpatrick, Swafford, & Findell; a publication from the U.S. Center for Education) argued that mathematics learners should have a 'productive disposition', which 'refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics' (p. 131). While this insistence on seeing sense seems reasonable in the context of well-documented, rule-driven or procedural activity, in which, ironically, 'making sense' is hardly operative, we can also read this as a new inscription device that goes even further and deeper in demarcating, dividing and excluding because of its focus on the individual. It is psychological in the sense that the child's soul is now at stake: '[P]sychological inscriptions focus on the interior dispositions or the *soul* of the child, fabricating the problem-solving child as a particular human kind for pedagogical intervention' (p. 4). Instead of talking about a child's ability to replicate or demonstrate understanding, *Adding it Up* invokes senses, beliefs and identity. There is now a new version of the aesthetic regime at play, in which the alignment between two kinds of senses – making sense, as in being understandable (epistemologically), and having sense, as in being worthy or useful (axiologically) – are what sustains consensus.

As mathematics teachers, we also find ourselves hoping that learners see value in the mathematics they must ~~take~~ in school. We hope this in part because of our own rich and satisfying experiences with mathematics. We also realize, following Bishop (1988), that mathematics carries with it a particular set of cultural values that are most often not made explicit in the mathematics classroom. However, these values are much more complex than 'useful and worthwhile'. Indeed, most of mathematics is *not* useful. Steady effort does not always pay off. Being an effective learner and doer of

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mathematics might conflict with other ethical commitments. Mathematics is rife with nonsense. Instead of coercing students to embrace the new, more all-encompassing aesthetic regime (an effort unlikely to succeed),[6] why not see what the reigning mathematical aesthetic offers as opportunities for dissensus? Consider the learner who manages to make sense of Pick's theorem while also coming to see it as utterly useless and trivial; or ~~consider~~ the learner who perceives the usefulness of 'invert and multiply' when dividing fractions but will not make sense of it.

The psychologies of instruction that are invented by reform (be it a two-column proof or problem solving) are designed to normalize the child, so Popkewitz (2004) argues that they are inadequate for the purposes of translating mathematics into curriculum projects. He suggests that translation requires intellectual tools that 'consider the relation between the knowledge (concepts, generalisations) and the cultural practices that enable the production of the knowledge' (p. 27) in such a way as to avoid psychological reductionism. Insofar as Lakatos's *rational reconstruction of history* (his term) focuses on relations that construct the discipline and exemplifies the way in which the discipline grows and changes over time, he can be seen as offering tools for thinking about and ordering the practices of mathematics. One of the important features of mathematical growth to which Lakatos points relates to the way in which 'taste' plays an important role in determining the shape of the proving process. Interestingly, this feature is completely absent in any pedagogical translation of Lakatos's work. And it is arguably one of the crucial points at which the vision of a discussion-based, problem-solving reform classroom breaks down: How does one decide, in the absence of any purely logical means, which definition will be chosen, which claims will be embedded into lemmas, which mathematical monsters will be explained away?

As Popkewitz and many other scholars have persuasively argued, schooling is a form of cultural politics, whereby particular social agendas become entrenched as common sense. School mathematics – as a 'high-status discipline' (Ahlquist, 2001, p. 27) – plays a significant role in the production and validation of what is taken to be common sense. The extent to which one masters the alphanumeric practices associated with this common sense correlates in large part with the acquisition of cultural capital. This

[6] While research has shown correlations between belief in the worthiness and usefulness of mathematics and strong achievement, as well as between self-efficacy and strong achievement, it is extremely misguided to assume that there is a causal relationship or that is it known how such a 'productive disposition' might be taught or instilled in the mathematics classroom.

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cultural capital is distributed according to socio-economic class, 'race'/ethnicity, gender, (dis)ability and other social categories (Gates, 2002). As a 'critical filter' implicated in the social stratification of communities, school mathematics becomes pivotal in the social structuring of students' lives (Moses & Cobb, 2001).

Skovsmose (1994) uses the term 'critical mathematics education' to describe attempts to address this fact and to reconceive school mathematics as a site of political power and ethical contestation. Various proponents of critical mathematics education have pursued this agenda in different ways.[7] Skovsmose and Borba (2004) are careful to suggest that the critical approach must always attend to the 'what if not' of school mathematics – that we must investigate the possible, consider the otherwise and explore 'what could be' (p. 211). ~~They argue that researchers and educators must imagine alternatives that trouble the current situation by actively and creatively generating visions or descriptions of a mathematics education that are more inclusive, more artful, more full of surprise.~~ This approach 'confronts what is the case with what is not the case but what could become the case' (Skovsmose & Borba, 2004, p. 214). Similarly, Pimm (1993) encourages a shift 'from should to could' in teacher education, arguing that the lust for change (which focuses on how teachers should teach and students should learn) ignores the sense of the personal and the possible. Given that schooling is a form of cultural politics, we need to study mathematics sense-making in terms of a 'distribution of the sensible'. Our inclusive materialism is an attempt to do so.

In this chapter, we have pointed to several different acts of dissensus, beginning with that of Sean, who tried to shift the binary distinction of even and odd numbers into a tripartite one, in which numbers could be even, odd, or both even and odd. We also elaborated on how surprise and counter-examples entailed dissensus by significantly shifting our ways of sensing, thus calling into question what makes sense. Even if Sean's classmates and teacher had joined his revolution, a new regime would have inevitably settled in (albeit an interesting one, where, say, rectangles might be both squares and parallelograms or quadratic functions might

[7] For example, by generating a socio-political ethics of mathematics education (Skovsmose & Valero, 2002; Valero, 2004; Valero & Zevenbergen, 2004); designing new mathematics curricula that address social justice issues (Mukhopadhyay & Greer, 2001; Gutstein, 2006; Tate, 2005); examining mathematics teacher identity and resistance to social justice pedagogy (de Freitas, 2008a de Freitas, 2008b; Rodriguez & Kitchen, 2005; Walshaw, 2004a Walshaw 2004b; Zevenbergen, 2003); and deconstructing the linguistic strategies unique to school mathematics that inhibit increased participation (Adler, 2001; de Freitas & Zolkower, 2009; Morgan, 2006).

be both first and second degree). As consensus grows, every shocking new counter-example (the sphere as a polyhedron!; the everywhere continuous but nowhere differentiable function!) gets tamed into the ‘obvious’ and the ‘clear’. When Brown and Walter (1983 advocated for problem posing in the mathematics classroom, they offered the radical, and surprising, idea that mathematics could also involve posing problems, not just solving them~~, and that students could do this, too~~. Allowing students to pose problems, just like mathematicians do, changed the nature of who was in control of the questions in the classroom. But it turned out to be rather hard to manage at scale. So, in the first step towards consensus, the ‘what-if-not?’ means of posing problems was offered to teachers and students, but it was all too often realized, unfortunately, as a set of prescribed alterations to a given problem situation. Then, most ironically, researchers began to study whether problem posing could work in the service of other (more desirable) goals, so that the very idea of posing problems folded into an existing consensus around school mathematics.

Dissensus eventually produces a new consensus. The question thus becomes: How might dissensus-producing ideas be kept lithe and fleeting, so that they escape becoming part of the common sense while remaining meaningful for a community of practice? We have offered surprise (which is often produced by counter-examples) as a sign that there is opportunity for much-needed dissensus, one that has a progeny within the discipline of mathematics. Surprise is often short-lived. It often occurs at the local level and is produced by disturbing expectations, be they mathematical or pedagogical: for example, by asserting that a line is a circle with the centre far, far away; by imagining that odd numbers do not exist; by teaching quadratic functions before linear ones or tangent before sine and cosine; by letting parallel lines meet; by asking for a wrong answer with an explanation; by showing how judgements of simplicity are driven by context; etc. The aim is to perturb, if even only temporarily, what is taken to be common sense and who is assumed to possess it.

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