Big Mathematical Picture (BMP) Commentary

AIM-TRU

Analyzing Instruction in Mathematics using the Teaching for Robust Understanding Framework

How to Use the BMP Commentary

This document is designed to be used as a starting point for facilitators to anticipate Big Mathematical Pictures (BMPs) that teachers may discuss during this part of the PD cycle.
Each page of this resource contains a catalog of BMPs that have been generated by prior mathematics teacher communities for specific Formative Assessment Lessons arranged by the grade level teaching assignments of the community. This is not intended to be an answer key for facilitators to use to determine if participants' BMPs are correct, but rather a window into insights that other communities have generated over time. New BMPs outside of the ones listed are certainly possible and probable.
Prior to facilitating, we suggest reading through past BMPs to assist if your community struggles with articulating their own.
We intend to update this BMP Commentary often as new communities discover novel ways to describe the BMPs for specific Formative Assessment Lessons.

Example of One Community's Development of the Big Mathematical Picture

Please feel free to contact our team at AIMTRUinfo@gmail.com with any questions you have about the BMP Commentary or how you might use it in your setting.

AIM-TRU

Table of Contents

AIM-TRU

Analyzing Instruction in Mathematics using the Teaching for Robust Understanding Framework

Applying Properties of Exponents

Group	BMP Outcomes
Mixed (Middle School & High School)	Equivalence of expressions, representations via writing and simplifying ("breaking down numbers and building them back up")
	Writing equivalent representation helps find underlying connections
	Generalizing from examples - using structure
	Verifying through computation
	Building efficient notation
	Express different forms
	Equivalence as a problem solving tool
	How can rewriting help us problem solve?
	Why one representation can be more useful
	What does an exponent mean?
	Exploration to lead to generalization
	When can I use the rules/laws?
	What is the relationship between all of these operations?
	Equivalence (How can things look different and have the same value?)
	Memorizing vs understanding
	Applying properties to generate equivalent expressions
	Efficiency
	Connection between rules (When do I use each one? When is this rule the best?)
	Understanding structure
	Extension to higher exponents
	Why do these properties make sense?
Middle School	Order of operations
	Efficiency
	Equivalent representations
	Defending one's mathematical thinking
	Representing and connecting
	Exponents operate on properties/rules; Recognizing forms (as students advance, they start to see why structure works and why it's important); The value of standard forms in seeking equivalent expressions
	Properties enable us to efficiently manipulate expressions and connect different equivalent representations and recognize important forms
	Students understanding of exponents opens up layers to understand relationships to efficiently evaluate mathematical problems
	Converting to different bases
	Different strategies to justify if the cards were equal
	Equivalent ways to write different expression
	Expressions were written in Desmos in a more mature mathematical manner
	Communication about mathematical ideas
	Justify equivalence by using laws of exponents
	How are the cards equivalent?

Back to TOC

Comparing Lines and Linear Equations

Group	BMP Outcomes
Mixed (Middle School & High School)	Calculating slope
	"What do I need to solve?"
	Interpreting graphs
	Connecting all representations & objects
	Each representation can illuminate the others
	Some representations can be more useful than others
	The scale changes things
	Slope as a proportional relationship
	Different representations - being able to see connections between diagrammatic, graphical, and algebraic representations
	Relationship between variables - independent, dependent. How do x and y change with each other?
	Idea of an inverse - as one is changing the other is changing equally
	Math can be used to describe real situations - measuring in time is something that is used
Middle School	Understanding and justifying the connection between the interrelated nature of key aspects (slope, y-intercept, scale) of graphs, equations, and visual representations
	Understanding the relationship between a graph and its equation & understanding the relationship between related increasing and decreasing graphs
	Prior knowledge about slope and linear equations helps students to understand the changes that occur in a situation represented by graphs and equations
	Recognize the elements of the equation of lines in different context (conceptual) and make connections visually and graphically and with equations
	Slope as a numerical and visual representation of a proportional relationship
High School	Functions; Relating quantities to each other and using an equation to model this relationship
	Understanding slope and y - intercept allows us to see the graph more clearly, adds to mathematical literacy and understanding and seeing a need for mathematics
	Slope can be a connection between science and math
	Interpreting the details of a situation can give a clear picture
	Negative slope is an inverse relationship
	Slope/rate of change can be represented in many different ways.
	Slope is a unit rate of change found in graphs, equations, tables, models
	Rates of change quantify relationships between variables in graphs, equations, tables
	The most significant idea is that multiple representations are important because it allows students to have multiple ways to understand the material
	Slope as a unit rate of change
	Slope measures a linear relationship between independent and dependent variables and can be used to interpret real world situations

Back to TOC

Comparing Strategies for Proportion Problems

Group	BMP Outcomes
Mixed (Middle School & High School)	Actually measuring
	More than one way to solve
	Connecting methods will shine a light on the most efficient method
	Idea of equivalence
	Connecting geometry to something that is not geometric
	Equivalency by experience
Middle School	Students need to understand that “scaling up” or “scaling down” creates equivalence, just in larger or smaller quantities
	Students must understand that scaling is a multiplicative process. Students must understand that a proportional relationship has a consistent pattern or a constant that describes it Scale factors
	Having a good understanding of basic math skills to explain when a qu antity should scale up or down
	Understand that scaling can be done by multiplication or division of the same scale to get equivalent values
	Fractions, equivalent fractions, unit rate, ratios, part to part and part to whole ratios
	A proportional relationship is multiplicative whether increasing or decreasing
	Fluidity with number-sense...ratios, proportions, scaling, etc.
	Real world problems involving proportional relationships, scale factor and unit rate, can be solved in different ways, some more efficient than others
	Constant of proportionality/slope and how it relates to finding unit rate
	How these ideas relate to real life
	Equivalence (fractions; multiple solutions to find the same answer)
	The meaning of multiplication as scaling and the relationship to division, percentages
High School	Equivalency and ratios - what stays the same, and what changes?
	Why and how can I to use multiplicative thinking (verus additive thinking) when solving a problem?
	“What is multiplication?”
	What factors help determine the efficiency of a method?
	Scale factor/unit rate/proportionality
	Estimation - how can we determine a relationship/pattern based on given evidence?
	Proportional reasoning - multiplying by a unit rate, scaling up/down
	Unit rate
	Using the unit rate to solve
	How can we use a scale factor and its reciprocal as an alternative to writing out proportions?
	Scaling up/down using a unit rate

Back to TOC

Describing and Defining Quadrilaterals

Group	BMP Outcomes
Mixed (Middle School & High School)	Sets and relationships (understanding that there are a limited number of necessary properties to define a quadrilateral; properties, characteristics, and definitions - can these be the same?)
	Vocabulary (use of vocabulary and symbols; developing visual skills/visual vocabulary)
	Congruency (similarity vs congruence)
	Logical reasoning
Middle School	Using precise vocabulary
	Connecting multiple representations
	Shapes are collections of properties
	Minimal properties needed to determine a quadrilateral
	Certain properties allow us to classify shapes, the distinct properties become defining
	Defining characteristics are unique to the properties of specific shapes
	Recognize the overlapping characteristics of shapes as well as their specific identifying characteristics
	A set of specific, minimal properties determine a shape, and other properties are determined by that set of properties
High School	The most efficient approach involves using the properties that encompass the most properties
	Properties are powerful and not necessarily defining; Adding and subtracting them can lead to construction of a particular quadrilateral
	Properties of lengths and angles are not stand alone; they are connected
	Efficiency - starting with the most basic characteristics and adding only those needed to construct a particular figure
	A quadrilateral ‘bridge’
	Counterexamples are also powerful
	Construct a valid argument
	All quadrilaterals can be identified using a minimal number of characteristics
	Students can used properties of quad to visually represent it
	Classify
	2D shapes can be classified, drawn, related to, contrasted with each other, using a minimal number of characteristics/properties
	Relationships between shapes
	Finding a relationship in order to classify

Back to TOC

Interpreting Distance-Time Graphs

Group	BMP Outcomes
Mixed (Middle School & High School)	Positive, negative, and constant slopes in context
	How distance and time relate to each other
	Slope = speed
	The discussions and connections to real life situations help the math come to life
	Distance from home vs just distance
	Instantaneous changes in slope - A change from positive to negative slope means they never stop and change direction instantaneously
	What is the meaning of a slope (Curves vs lines, and what a change in slope means)
	The relationship between two variables (as they relate to slope - a rate)
	The importance of a function - that they “break” the storyline
	Idea that a visual representation (graph) may be different than a visual representation (in your head). What does that mean about mathematical modeling in these instances?
	Comparative rates - what does fast, slow, positive, negative mean? (Relationship to functions; increasing vs decreasing functions)
Middle School	The relationships between distance, speed, and time can be represented in diagrams with matching scenarios; slope represents a relationship between quantities
	Abstract representations can tell a story, if you take the time to interpret it
	Different elements of graphs have real world meanings and can be connected to Algebraic representations
	Features of abstract representations (e.g. slopes in graphs) give key information about the situations they model (e.g. stories)
	Relationship between variables
	Story contexts relating to graphs (axes labels/meanings)
	Meaning of slope (rate of change)
	Extensions and applications to other disciplines
	Graphs are not pictures/maps of scenarios
	There are multiple equivalent ways to represent a situation involving rate of change through mathematical relationships and connections that allow for key features to be illuminated
	Represent a situation numerically, graphically, and in context - Less focus on slope
	Connections to proportional relationships
	Recognize key features of the graph/story and how that relates to the context
	Constant rate of change vs. variable rate of change would be represented on a graph vs. with context
High School	Comparing distance and time through engineering and reverse engineering through modeling
	Using words and the graph to connect to the slope
	Interpreting the picture as a distance time relationship
	Interpreting multiple representations of rates of change as means to make sense of a situation
	Connecting picture, story, and slope
	A real life connection to the slope
	Interpreting a graph as a relationship instead of a picture
	Capturing a real world situation using a picture
	Visually representing

Back to TOC

Interpreting Multiplication and Division

Group	BMP Outcomes
Mixed (Middle School & High School)	Use of visual models to represent the relationships
	Understanding of operations based on words
	Interpretation of expression and link to the visual
	Evidence of multiplication/division yielding the same solution (inverses) - connection to fact families
	Number sense through the written description of the solutions, but before what is the problem asking?
	There are different ways to model a situation
	There are many equivalent representations to model a situation
	Equivalence - Strengthened and justified through multiple representations
	Do words in English lead to unique mathematical expressions, or, much like English, does math have several ways of saying the same thing?
	The big idea is modeling different situations using rational numbers.
	Operation meanings and relationships
	Using visual modeling to represent equivalent expressions
	Big idea: the same number sentence can be associated with different concrete or real-world situations & different number sentences can be associated with the same concrete or real-world situation
	Interpreting different meanings of multiplication and division
	There multiple representations of fractional and whole number operations

Solving Linear Equations

Group	BMP Outcomes
Mixed (Middle School & High School)	Equivalence: different ways of writing equations, stories, different representations
	Grouping -> the meaning of parentheses
	Algebraically represent a real world situation -> strategies for modeling
	Increasing access with different representations
	Equivalence between forms of equations
	Equivalent representations (distributive property/all properties)
	Being able to go both directions (undoing/working backwards)
	Modeling (real world vs mathematical)
	Giving objects meaning
	Matching/connecting specific parts
	Changing vs unknown
	Interpretation of a situation
	What is x?
	P, Q, R as constants vs variables
	Generalization
Middle School	There is value in safe, productive, thoughtful math struggle in deepening understanding of equivalent representations
	Every line in your solution is an equivalent equation to everything that came before and after that line
	Understanding that real world contexts have different representations of equations that are equivalent and can be solved using different methods

Back to TOC

Evaluating Statements about Radicals

Group	BMP Outcomes
Mixed (Middle School & High School)	Guessing + Checking
	Solving/simplifying
	Problem solving strategies -> Specific techniques for the power exponent) 2, √
	Non-strategies (“What I can’t do”)
	Equality; maintaining equivalence while solving
	Applying prior knowledge to unfamiliar objects
	Equivalent expressions, solutions
	Identities and equivalence
	Understanding that proving a mathematical identity can not be done with examples, but proving that something is not a mathematical identity can be done with counterexamples
	Generalizing algebraic rule
	Proof
	What constitutes a solution
	“How do you prove an identity to be true?”
	What does a solution mean?
	Determining conditional equivalence vs. identities
	Equivalent radical expressions and their solutions
	Proving that two expressions are always equivalent
	Identity equations vs conditional equations
	Equivalence
High School	Don’t over-generalize; Use intuition, then use knowledge of equivalence to make a determination
	Algebraic manipulation to prove identities
	Determine equivalence and justify reasoning
	Intuition is like a hypothesis, and algebraic manipulations allows to prove to yourself
	Collaboratively investigate and communicate justification for equivalence/ non-equivalence of radical expressions

Interpreting Algebraic Expressions

Group	BMP Outcomes
Mixed (Middle School & High School)	Connection between different representations
	There are multiple ways to represent mathematical expressions: symbols, words, area model are some of representations
	Connecting the multiple representations of a mathematical expression and gaining fluency in navigating between them
	Equivalence: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value
	The idea of equivalence - that any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value
	Equivalent mathematical expressions can be represented in infinite ways and across multiple representations
	Precision in interpretations of symbols, words and visual representations
	Comparing equivalent expressions in various representations
	Attending to structure
Middle School	Algebraic distributive property connected to area model
	Equivalency (recognizing through various representations) Fluency/flexibility with the language of math
	Students explore the relationship between algebraic expressions, verbal expressions, area models, and tables; They recognize that an expression can be represented in multiple equivalent ways and students make connections between different forms of an expression
High School	Equivalence between multiple representations
	Equivalence: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value
	Equivalence of algebraic expressions can be represented in different forms: tables (inputs/outputs), expressions, verbal and written descriptions, as well as visual models
	Equivalence
	Multiple representations
	Order of operations
	An expression can be represented in several equivalent forms: verbal and algebraic expressions (with varied applications of the order of operations), tables and areas
	While there are infinite forms of equivalent written representations, there is one table and one total area
	There are infinite forms of algebraic representations and multiple ways to represent an expression verbally, but only one table and one total area for an expression

Back to TOC

Modeling Motion: Rolling Cups

Group	BMP Outcomes
Mixed (Middle School & High School)	A systematic approach when modeling
	How one quantity depends on another - How changing different variables affect each other
	Figuring out interactions
	Generalize something concrete, into something more abstract

AIM-TRU

Representing Linear and Exponential Growth

Group	BMP Outcomes
Mixed (Middle School & High School)	Linear and exponential relationships can be compared using various different but equivalent representations
	Different representations can model the same situation. We need different models to match different situations
	Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways It is necessary to switch throughout representations at times
	Exponential growth will eventually outpace linear growth
	Models are used to to make predictions
High School	Comparison of Growth vs Growth vs Growth Rate
	Comparison: tables, equations, graphs are used to compare simple and compound interest rates.
	Comparison: tables, equations, graphs are used to compare rates of change
	Comparison: tables, equations, graphs are used to compare rates of change which can be applied in real world situations
	Comparing and contrasting rates of change and the effect of varying components (principle, interest, and time) while using the structures of multiple representations

Representing Probabilities: Medical Testing

Group	BMP Outcomes
Mixed (Middle School & High School)	How to find conditional probability of an event A, given an event B
	Quantitative reasoning
	Independent vs. dependent events
	Mathematical modeling of sample spaces
	Identifying different approaches to solve the problem
	Practice standards: make sense of problems, modeling
	How can part/whole relationships be used to analyze and predict
	Identifying subsets and modeling relationships between sets
High School	Apply different tools to represent data and define universal in order to calculate the probability and interpret the result and make predictions for further real-life situation
	The value of probabilities are dependent on how “all possible outcomes” are defined
	All probabilities are connected to each other; everything affects one another
	“Additional conditions may change existing probabilities“
	Probability can be used to make predictions about real life situations and different representations can be used to arrive at that answer
	Visual representation of data allows us to represent data in many ways
	Understanding the mathematical connections between values and data sets across various representations to make sense of the given problem situation - Define the universe
	Probabilities, and in particular conditional probabilities, can be evaluated by taking the quotient of (the number of desired outcomes) and (the number of possible outcomes), where the outcome can vary depending on what is being asked for - the data can be portrayed in several different ways
	Solving complex real-life situations that involve sets and subsets, requires making sense of the problem (careful reading, checking thinking, taking time), communicating with others, using models to support sense making and precision in problem solving

Representing Quadratic Functions Graphically

Group	BMP Outcomes
Mixed (Middle School & High School)	Patterns and relationships through the various forms; equations look different but are still equivalent mathematically/visually
	The visuals and the equations both provide building blocks to the larger story (based on their components)
	Different representations (between graphs and equations)—how the equations are shown through the graphs of parabolas
Middle School	Equivalent forms of algebraic functions illuminate different key features of their graph
	Multiple representations for the same relationship
	The structure of the equation tells you something about the graph, and the graph can tell you something about the equation
High School	Functions have different forms of expression that are ultimately equivalent to one another and the structure and parameters provide important information about key features of their graphs
	The big mathematical idea is the connection between the structure of the equivalent equation and its graph

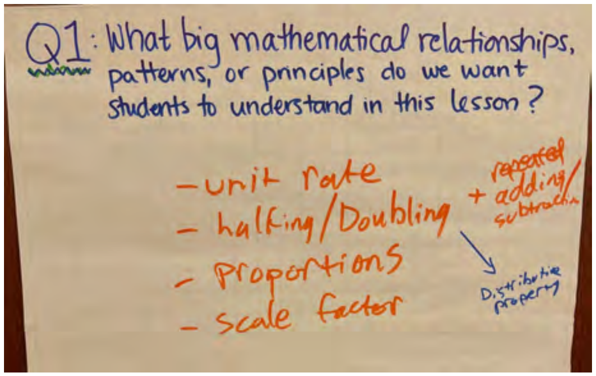
Solving Linear Equations in Two Variables

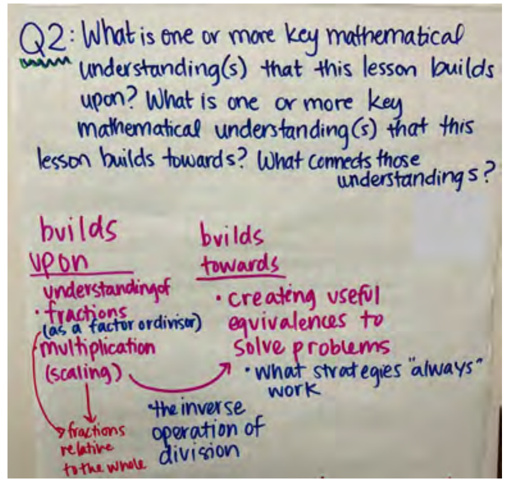
Group	BMP Outcomes
Mixed (Middle School & High School)	Be able to decontextualize a situation and model it mathematically and then find an equivalent mathematical model and solve
	Students need an understanding of equivalence and number properties, mathematical practices, and various representations
	Understanding and Interpreting the meaning of algebraic expressions based on a real-world context involving solving a system of linear equations with multiple variables in multiple ways and verifying the appropriateness of the solution
High School	Understanding the many pathways to viable solutions (elimination, substitution, graphing, guess & check) and the advantages of each
	Equivalence
	Equivalent equations solve applications by using various mathematical structures and different forms/representations

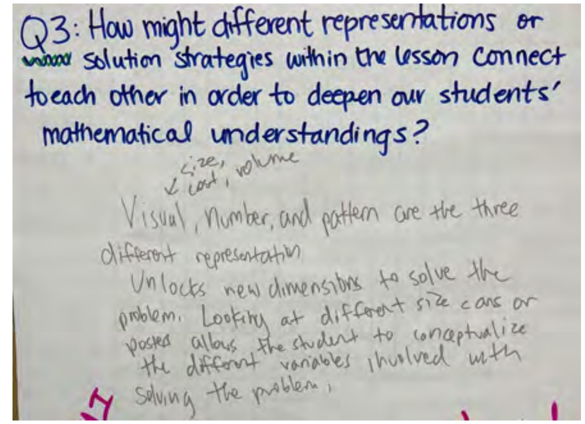
AIM-TRU

Example of a ig Mathematical Picture Discussion for the Applying Properties of Exponents FAL

1. In small groups, teachers answer and chart one of the following questions related to the FA . This happens after all teachers have initially read the FAL.







2. Teachers do the mathematical task for the FA . In this specific FAL, Applying Properties of Exponents, the mathematical task consisted of card sort activity in which students are asked to match single exponent cards with numerical expression cards.

E12	E10	S6	S10
$2^3 \times 2^3$	$(2^3)^2$	2^6	4^3

3. In the same small groups, teachers revisit their discussion related to the three questions and articulate a big mathematical picture:

Question	Big Mathematical Picture
What big mathematical relationships, patterns, or principles do we want students to understand in this lesson?	Proportional thinking can be represented using different numerical strategies.
What is one or more key mathematical understanding that this lesson builds upon? What is one or more key mathematical understanding that this lesson builds towards? What connects those understandings?	A proportional relationship has a constant multiplicative pattern that can be used to create equivalences.
How might different representations or solution strategies within the lesson connect to each other in order to deepen our students' mathematical understandings?	Equivalent ratios can be scaled to compare, contrast and scale different representations proportionally.