4D reverberator–based digital filters

M.T. Koussoulis
Dept. Computer Science
Bergen Com. College
Paramus, NJ 07652, USA

C.A. Coutras, G.E. Antoniou
Dept. Computer Science
Montclair State University
Montclair, NJ 07043, USA

Abstract—This research presents a circuit realization for four-dimension (4D) reverberator–based two-input, two-output all-pass structured discrete filters. The proposed 4D circuit realization requires, for its implementation, a minimum number of delay elements. As a result, the dimension of the state-vector, of the derived 4D state space model, is minimal and its 4D transfer function is characterized by the all-pass property. A lower-order thorough example is provided to demonstrate the asserted minimality in both the circuit, and state-space realizations.

I. INTRODUCTION

Substantive accomplishments, in developing systems capable of synthesizing high-dimensional data, have been made in recent years. In the scope of multidimensional (nD) systems and digital filters, a growing set of contributions have been made in areas of digital signal and image processing, five-dimension (5D) depth-velocity filtering, cognitive radio, radio astronomy, robot vision and imaging [1]–[5].

Several published contributions in the growing area of 4D filter system design have been reported recently. These research findings are as follows: 4D light field filters; 4D X-ray tomography; and 4D cardiac CT images. [6]–[10].

This research work considers the problem of minimal circuit and state–space realization for 4D reverberator–based digital filters, characterized by the all–pass property. The outcome produced are circuit, and state–space realizations comprised of a minimal number of delay elements, and dimensions respectively.

The essentiality in providing minimal realization emanates not only out of hardware requirements but also because sometimes non-minimal realizations often cause theoretical or computational difficulties, due to the absence of the fundamental theorem of algebra for polynomials with more than one–dimension (1D) [4].

Realizations of minimal circuit and state–space representation have received considerable attention, even for two–dimensional (2D) digital systems, because minimal realizations are not always attainable in closed form [11]. Generalized 4D minimal circuit and state–space realizations have been given for finite impulse response (FIR), lattice, ladder, and lattice–ladder structured digital filters [12]–[15].

In this publication, circuit and minimal state space realizations for 4D reverberator–based digital filters and systems are considered. These classic 1D filters were independently proposed by Schroeder–Logan, and S.J. Mitra, and are characterized by the all-pass property [16], [17].

II. TWO–PORT 1D SECTION

A two–port 1D circuit characterized by the all–pass property, Fig. 1., can be described by the equations [17]:

\[
\begin{align*}
Y_1 &= z^{-1}X_2 + \Delta Y_2 \\
Y_2 &= X_1 - \Delta z^{-1}X_2.
\end{align*}
\]

Equation (1) can be written as:

\[
\begin{align*}
Y_1 &= \Delta X_2 + \Delta^2 z^{-1}X_2 - \Delta^2 z^{-1}X_2 + z^{-1}X_2 \\
&= \Delta Y_2 + \Delta^2 z^{-1}X_2 + (1 - \Delta^2)z^{-1}X_2 \\
&= \Delta Y_2 + \Delta z^{-1}X_2 + (1 - \Delta^2)z^{-1}X_2.
\end{align*}
\]

Using (2), the output \( Y_1 \) (5), becomes

\[
Y_1 = \Delta X_1 + (1 - \Delta^2)z^{-1}X_2.
\]

Therefore the two modified two–port equations, \( Y_1 \) and \( Y_2 \), are given below. The corresponding circuit implementation is depicted in Fig. 2.

\[
\begin{align*}
Y_1 &= \Delta X_1 + (1 - \Delta^2)z^{-1}X_2 \\
Y_2 &= X_1 - \Delta z^{-1}X_2.
\end{align*}
\]

The above equations, in matrix form, can be written as:

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
\Delta & (1 - \Delta^2)z^{-1} \\
1 & -\Delta z^{-1}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}.
\]

In the very early 1960’s, M.R. Schroeder and B.F. Logan proposed an equivalent circuit implementation, to Fig. 2., based on the reverberation principle with negative feedback, as depicted in Fig. 3., [16].

Fig. 1: A basic 1D two–multiplier all–pass section.
III. 4D CIRCUIT AND STATE SPACE REALIZATIONS

Extension of the previously discussed 1D results yields a new 4D digital filter with lattice structure as is shown in Fig. 3., following the theory of 1D and 2D lattice filters [3], [17].

The proposed generalized circuit utilizes $4n$ delay elements, and $4n$ multipliers, as shown in Fig. 4., where the number of delay elements are minimal. For a single 4D section, where $n = 1$ in Fig. 2., the number of delay elements is four, which is minimal.

The subsequent goal is the derivation of the 4D generalized state–space model $(A, b, c', d)$ having the well known Givone–Roesser structure, extended to 4D, with cyclic dimensional arrangement with respect to the four independent variables: $x_1^i(i, j, k, l), x_2^i(i, j, k, l), x_3^i(i, j, k, l), x_4^i(i, j, k, l)$ [18],[19]. To acquire the state–space equations for the 4D model, $(A, b, c', d)$, the following procedure is applied:

- Use the circuit representation, depicted in Figs. 5, 4.

- Label the outputs of the delay elements $z_{1}^{-1}, z_{2}^{-1}, z_{3}^{-1}, z_{4}^{-1}$ that correspond to the states of the model.

- Write, by inspection, one state equation for every delay element $z_{1}^{-1}, z_{2}^{-1}, z_{3}^{-1}, z_{4}^{-1}$.

- Rearrange the equations to have blocks of the state variables: $x_1^i, x_2^i, x_3^i, x_4^i$.

- Generalize the results.

Following the above procedure the generalized 4D state space matrix–vectors $(A, b, c', d)$ and the scalar $d$, are derived. The overall 4D Givone–Roesser state–space–model having cyclic structure, with respect to the variables, is given below [18], [19]:

\[
\dot{x}(i, j, k, l) = Ax(i, j, k, l) + bu(i, j, k, l) \tag{8}
\]

\[
y(i, j, k, l) = c'x(i, j, k, l) + du(i, j, k, l) \tag{9}
\]

where,

\[
x(i, j, k, l) = \begin{bmatrix}
  x_1^i(i, j, k, l) \\
  x_2^i(i, j, k, l) \\
  x_3^i(i, j, k, l) \\
  x_4^i(i, j, k, l)
\end{bmatrix}
\]

\[
y(i, j, k, l) = \begin{bmatrix}
  x_1^{i+1}(i, j, k, l) \\
  x_2^{i+1}(i, j, k, l) \\
  x_3^{i+1}(i, j, k, l) \\
  x_4^{i+1}(i, j, k, l)
\end{bmatrix}
\]

The matrices $A, b, c', d$ of the above 4D state–space model, are given below having the following dimensions respectively: $4n \times 4n, 4n \times 1, 1 \times 4n$.

Applying the 4D $z$–transform on (8,9), its corresponding 4D transfer–function takes the following form:

\[
T(z_1, z_2, z_3, z_4) = c'[Z - A]^{-1}b \tag{10}
\]
the direct sum.

space models of Givone–Roesser and cyclic, both extended to $\mathbb{Z}$

where,

A. First–order 4D digital filter
given in Fig. 4 to be

$\dot{y}(i, j, k, l) = Ax(i, j, k, l) + bu(i, j, k, l)$ (11)
y(i, j, k, l) = $c'x(i, j, k, l) + du(i, j, k, l)$ (12)

where,

$x(i, j, k, l) = \begin{bmatrix} x_1(i + 1, j, k, l) \\ x_2(i, j + 1, k, l) \\ x_3(i, j, k + 1, l) \\ x_4(i, j, k, l + 1) \end{bmatrix}$

The procedure to facilitate the traversal between the state–space models of Givone–Roesser and cyclic, both extended to 4D, is provided in [19].

IV. EXAMPLE

A. First–order 4D digital filter

Considering the output of the first–order circuit realization given in Fig. 4 to be $y(i, j, k, l)$, the corresponding 4D state–space realization takes on the following form:

$\dot{x}(i, j, k, l) = Ax(i, j, k, l) + bu(i, j, k, l)$ (11)
y(i, j, k, l) = $c'x(i, j, k, l) + du(i, j, k, l)$ (12)

where,

$A = \begin{bmatrix} -\Delta_1 \Delta_2 & 1 - \Delta_2^2 & \cdots & 0 & 0 & 0 \\ -\Delta_1 \Delta_3 & -\Delta_2 \Delta_3 & 1 - \Delta_3^2 & 0 & 0 & 0 \\ -\Delta_1 \Delta_4 & -\Delta_2 \Delta_4 & -\Delta_3 \Delta_4 & 1 - \Delta_4^2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\Delta_1 \Delta_{4n-1} & \cdots & \cdots & -\Delta_{4n-2} \Delta_{2n-1} & \cdots \\ -\Delta_1 \Delta_{4n} & -\Delta_2 \Delta_{4n} & -\Delta_3 \Delta_{4n} & -\Delta_4 \Delta_{4n} & 1 - \Delta_{4n}^2 \end{bmatrix}$

$b = \begin{bmatrix} \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \cdots \\ \Delta_{4n-1} \\ \Delta_{4n} \\ 1 \end{bmatrix}$

c' = [1 - $\Delta_1^2$ 0 0 0 ]

d = $\Delta_1$.

Using (10) the corresponding 4D transfer function $T_{1D}(z_1, z_2, z_3, z_4)$ of (11, 12) is presented in Table 1.

It is noted that in the 4D transfer function of Table 1, the denominator polynomial is the mirror–image polynomial of the numerator and vice–versa.

B. First–order 2D digital filter

The derived 4D reverberator–based transfer function Table 1, can provide all–pass structural information for low dimension filters. For example, if $V_1 = 1$ and $V_i (\forall i = 2, \cdots, 5 = 0)$ and making the delay elements $z_3, z_4$ not present in the transfer function of Table 1, the corresponding 2D transfer function has the following form:
In the above 2D transfer function the all–pass property, is evident.

Furthermore for the classical 1D case (Fig. 2.) the above 2D transfer function (13) takes the form:

$$T_{1D} = \frac{\text{Numerator}}{\text{Denominator}} = \frac{N(z_1, z_2)}{D(z_1, z_2)}.$$  (13)

The all–pass property on the above 1D transfer function is obvious.

V. CONCLUSION

Circuit and state–space realizations for 4D reverberator–based digital filters, are discussed in this article. The realizations proposed are presented with an absolutely minimal number of delay elements and with the dimension of the state–space vector being absolutely minimal, both being of $(4n)$. The number of multipliers is $12n$ and adders are $8n$ of the circuit realization. The 4D transfer functions of the realizations are characterized by the all–pass property, as in the classical digital signal processing filtering, and it is evident from examples provided.

REFERENCES


