5D Moog Ladder Structured Digital Filter: Minimal State-Space Realization

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***Abstract*—This paper outlines the minimal circuit and state–space realization for a five-dimensional (5D) ladder digital filter having the one-dimensional (1D) Moog structure and characteristics. The number of delay units and the state vector in the state–space realization are both minimal. Furthermore, in this work two first–order examples demonstrate the filter's internal structures by using simple state–space models. The first example employs a 5D circuit implementation, whereas the second is a two–dimensional (2D) filter. It is clear that the results present the minimal state–space realizations, as well as their transfer functions, which correspond to the derived generalized 5D Moog transfer function.**

***Index Terms*—Multidimensional systems, Moog filter, 5D filter, ladder structure, transfer function, minimal realization, circuit implementation, state–space.**

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I. Introduction

In the 1960's after the introduction of the renowned filter by R. A. Moog (1934-2005), it was effectively implemented in musical instruments, including filters, mixers, and controlled modular synthesizers. New musical sounds were generated as a result of the synthesizer's invention, which provided an extraordinary boost to the music industry. An evident feature of the block diagram structure of the Moog filter is the operation of two negative feedback loops. Both loops incorporate the delay unit of the filter structure [1], [2].

The presented work employs a cascade structure to expand the one–dimensional (1D) ladder Moor filter to a discrete five–dimensional (5D) setting. A minimal state–space realization for the 5D digital filter structure is derived using a cyclic 5D state–space model using the minimum circuit realization. The minimal nature of the realization is a highly essential factor in higher–dimensional filters or systems since the fundamental theorem of algebra is not valid [3]. So, minimal state–space realizations with the same size as the number of delay units are, at least in theory, appealing. This is because non–minimal implementations could cause design issues while making hardware more complicated.

Until today the present day, minimal state–space realizations for 5D filters and systems have been proposed for infinite impulse response (IIR) filters, all–pole lattice, as well as for conventional lattice finite impulse response (FIR) digital filters [4], [5].

The investigation of high–dimensional filters and systems is characterized by several idiosyncrasies, stemming not only from the absence of the fundamental algebra theorem [3], but also from the immense computational demands associated with all stages of the investigation, which is beyond human capabilities. Therefore, each research endeavor that yields a valuable outcome in this area of study significantly enhances the comprehension of the behavior of these systems.

The remaining manuscript is structured as follows: In section II, 1D Moog ladder digital filter is outlined, and in section III, the 5D state–space model realization is given. In section IV, the 5D Moog ladder structured digital filter is presented. Section V, discusses first–order five–dimensional and two–dimensional illustrated examples. The conclusion is provided in section VI.

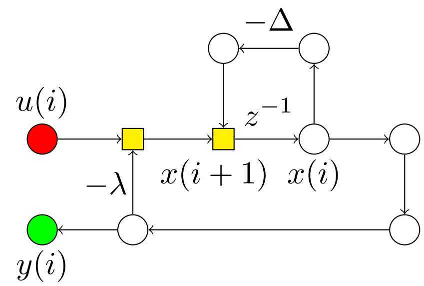

Fig. 1: Single–section 1D Moog ladder structured digital filter

II. 1D Moog ladder digital filter

The first–order one–dimensional (1D) ladder digital filter of Moog type, as depicted in Fig. 1, can mathematically be described by the following set of state–space and output equations:

$$x(i+1) = u(i) - \Delta x(i) - \lambda x(i), \quad (1)$$

$$y(i) = x(i), \quad (2)$$

where, $u(i)$ is the input, $y(i)$ is the output. z^{-1} is a time delay–unit. Δ, and λ are feedback coefficients.

Utilizing the 1D z–transform, the associated transfer function for the aforementioned state–space system (1), (2) is:

$$\frac{Y(z)}{U(z)} = \frac{1}{\lambda + (\Delta + z)}. \quad (3)$$

Additionally, the preceding 1D Moog ladder filter, Fig. 1, will be expanded to 5D in the subsequent section. Also, the generalized state–space model and its transfer function will be derived from the expanded 5D circuit realization.

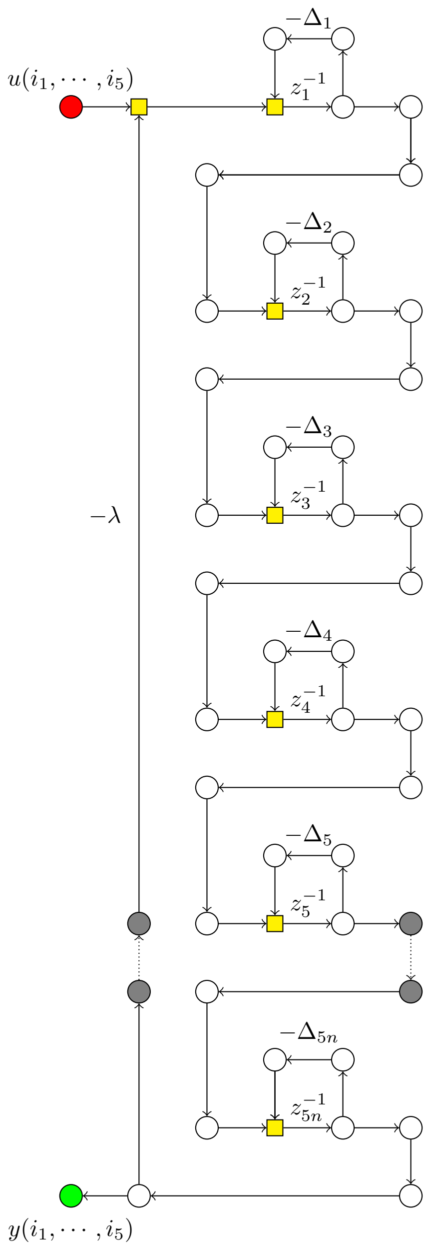

Fig. 2: Generalized 5D Moog ladder digital filter

III. 5D STATE–SPACE MODEL REALIZATION

The cyclic 5D state–space model used, is expressed by the following set of equations [6], [7]:

$$\dot{\mathbf{x}}(i_1, \cdots, i_5) = \mathbf{A}\mathbf{x}(i_1, \cdots, i_5) + \mathbf{b}u(i_1, \cdots, i_5) \quad (4)$$

$$y(i_1, \cdots, i_5) = \mathbf{c}'\mathbf{x}(i_1, \cdots, i_5) \quad (5)$$

The dimensions of the 3 matrices $\{([\mathbf{A}], [\mathbf{b}], [\mathbf{c}'])\}$, and the scalar d of the aforementioned state–space model (4), (5), that describe the digital filter, are: $[\mathbf{A}] \to (5n \times 5n)$, $[\mathbf{b}] \to (5n \times 1)$, $[\mathbf{c}'] \to (1 \times 5n)$.

The vectors $\dot{\mathbf{x}}(i_1, \cdots, i_5)$, $\mathbf{x}(i_1, \cdots, i_5)$, (4) and (5), of the 5D model have the following structures:

$$\mathbf{x}(i_1, \cdots, i_5) = \begin{bmatrix} x_1^{d_1}(i_1, \cdots, i_5) \\ x_1^{d_2}(i_1, \cdots, i_5) \\ x_1^{d_3}(i_1, \cdots, i_5) \\ x_1^{d_4}(i_1, \cdots, i_5) \\ x_1^{d_5}(i_1, \cdots, i_5) \\ \cdots\cdots\cdots\cdots \\ x_{5n}^{d_1}(i_1, \cdots, i_5) \\ x_{5n}^{d_2}(i_1, \cdots, i_5) \\ x_{5n}^{d_3}(i_1, \cdots, i_5) \\ x_{5n}^{d_4}(i_1, \cdots, i_5) \\ x_{5n}^{d_5}(i_1, \cdots, i_5) \end{bmatrix}, \quad (6)$$

$$\dot{\mathbf{x}}(i_1, \cdots, i_5) = \begin{bmatrix} x_1^{d_1}(i_1+1, i_2, \cdots, i_5) \\ x_1^{d_2}(i_1, i_2+1, \cdots, i_5) \\ x_1^{d_3}(i_1, i_2, i_3+1, i_4, i_5) \\ x_1^{d_4}(i_1, \cdots, i_3, i_4+1, i_5) \\ x_1^{d_5}(i_1, \cdots, i_4, i_5+1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{5n}^{d_1}(i_1+1, i_2, \cdots, i_5) \\ x_{5n}^{d_2}(i_1, i_2+1, \cdots, i_5) \\ x_{5n}^{d_3}(i_1, i_2, i_3+1, i_4, i_5) \\ x_{5n}^{d_4}(i_1, \cdots, i_3, i_4+1, i_5) \\ x_{5n}^{d_5}(i_1, \cdots, i_4, i_5+1) \end{bmatrix}. \quad (7)$$

The transfer function is obtained by performing the 5D z–transform directly on (4), (5).

$$T_F(z_1, \cdots, z_5) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b}, \quad (8)$$

where, $\mathcal{Z} = z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n \oplus z_3\mathbf{I}_n \oplus z_4\mathbf{I}_n \oplus z_5\mathbf{I}_n, \cdots, \oplus z_5\mathbf{I}_n$, with $\oplus$ denoting the direct sum.

Using the 5D ladder Moog structured digital filter with minimal delay–units, the corresponding 5D cyclic state–space realization with minimal dimension is derived below.

IV. 5D MOOG LADDER STRUCTURED DIGITAL FILTER

Figure 2, shows the generalized 5D circuit implementation, in block diagram formulation, of a 1D Moog ladder structured digital filter as given in [1], [2]. The circuit implementation includes $(5n+1)$ multipliers and adders, in addition to $(5n)$ delay units. Using the generalized circuit implementation (Fig. 2), the mathematical formulation of the 5D state–space model $\{([\mathbf{A}]; [\mathbf{b}]; [\mathbf{c}']; d)\}$ is derived. The state–space equations for the 5D model $\{([\mathbf{A}]; [\mathbf{b}]; [\mathbf{c}']; d)\}$ may be obtained by the following algorithmic steps: "Label the delay–unit outputs (in Figs. 2 and 3) to identify all the states of the circuit. By examination, generate a state equation for every delay–unit, following the circuit theory analysis. Organize the equations such that every block comprises all of the available state variables. Expand the findings", as was stipulated in [4], [5].

As a result, the matrices that represent the obtained 5D structured state–space model (4), (5) have dimensions: $\{([\mathbf{A}]; [\mathbf{b}]; [\mathbf{c}']; d)\}$, the dimensions are: $[\mathbf{A}] \to (5n \times 5n)$, $[\mathbf{b}] \to (5n \times 1)$, $[\mathbf{c}'] \to (1 \times 5n)$, and finally are presented

$$
\mathbf{A} = \left[\begin{array}{ccccccccc}
-\Delta_1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & -\lambda \\
1 & -\Delta_2 & 0 & & & & & & \vdots \\
0 & 1 & -\Delta_3 & 0 & & & & & \vdots \\
0 & 0 & 1 & -\Delta_4 & 0 & & & & 0 \\
0 & & 0 & 1 & -\Delta_5 & \ddots & & & 0 \\
0 & & & 0 & 1 & \ddots & \ddots & & 0 \\
\vdots & & & & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 1 & -\Delta_{5n}
\end{array}\right], \quad \mathbf{b} = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right],
$$

$$
\mathbf{c}' = \left[\begin{array}{ccccccccc} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{array}\right].
$$

5DgM: Generalized 5D Moog ladder digital filter, state–space realization model

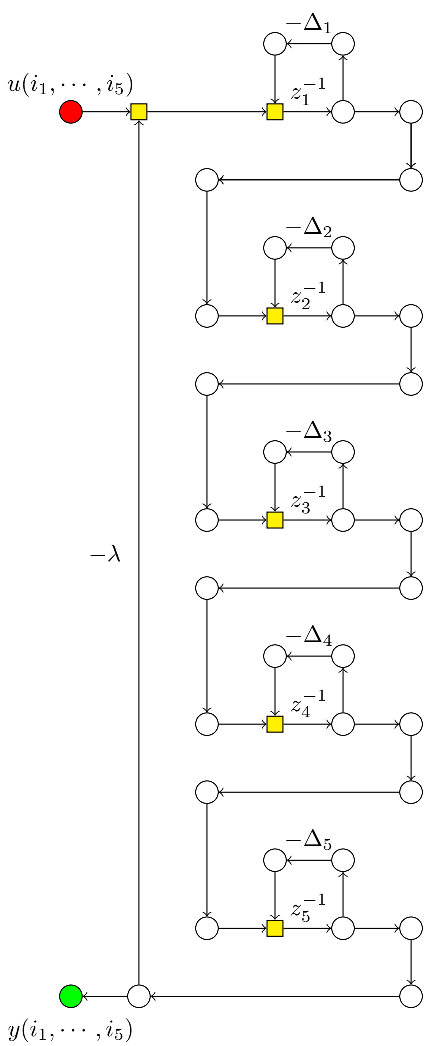

Fig. 3: 5D Moog ladder structured digital filter

in 5DgM (top, next page). Note that the order of the filter is closely proportional to the dimension of the model.

Using the 5D state–space equations (4), (5) with (8) and the state–space matrices 5DgM, the generalized 5D transfer function is:

$$
T_{5D(g)}(z_1, \cdots, z_5) = \frac{1}{\lambda + \prod_{i=1}^{5n} (\Delta_i + z_i)}. \qquad (9)
$$

Two first–order examples which illustrate the filter, are one in five dimensions and the other in two, and will be shown in the section that follows.

V. Examples

A. 5D, First–order Moog ladder digital filter

The state–space formulation in 5D, related with the output $y(i_1, \cdots, i_5)$ of the first–order 5D circuit implementation depicted in Fig. 3, can be expressed as follows [6], [7]:

$$
\begin{aligned}
\dot{\mathbf{x}}(i_1, \cdots, i_5) &= \mathbf{A}\mathbf{x}(i_1, \cdots, i_5) + \mathbf{b}u(i_1, \cdots, i_5), \quad (10) \\
y(i_1, \cdots, i_5) &= \mathbf{c}'\mathbf{x}(i_1, \cdots, i_5), \quad (11)
\end{aligned}
$$

where,

$$
\mathbf{x}(i_1, \cdots, i_5) = \left[\begin{array}{c}
x_1^{d_1}(i_1, \cdots, i_5) \\
x_1^{d_2}(i_1, \cdots, i_5) \\
x_1^{d_3}(i_1, \cdots, i_5) \\
x_1^{d_4}(i_1, \cdots, i_5) \\
x_1^{d_5}(i_1, \cdots, i_5)
\end{array}\right],
$$

$$
\dot{\mathbf{x}}(i_1, \cdots, i_5) = \left[\begin{array}{c}
x_1^{d_1}(i_1 + 1, i_2, i_3, i_4, i_5) \\
x_1^{d_2}(i_1, i_2 + 1, i_3, i_4, i_5) \\
x_1^{d_3}(i_1, i_2, i_3 + 1, i_4, i_5) \\
x_1^{d_4}(i_1, i_2, i_3, i_4 + 1, i_5) \\
x_1^{d_5}(i_1, i_2, i_3, i_4, i_5 + 1)
\end{array}\right],
$$

and,

$$\mathbf{A} = \begin{bmatrix} -\Delta_1 & 0 & 0 & 0 & -\lambda \\ 1 & -\Delta_2 & 0 & 0 & 0 \\ 0 & 1 & -\Delta_3 & 0 & 0 \\ 0 & 0 & 1 & -\Delta_4 & 0 \\ 0 & 0 & 0 & 1 & -\Delta_5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{c}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrices $\{([\mathbf{A}]; [\mathbf{b}]; [\mathbf{c}']; d)\}$ of the above model (10), (11) have dimensions: $[\mathbf{A}] \to (5 \times 5), [\mathbf{b}] \to (5 \times 1), [\mathbf{c}'] \to (1 \times 5)$.

Utilizing the 5D z–transform (8) on equations (10) and (11), the corresponding transfer function in five dimensions $T_{5D(1)}(z_1, \cdots, z_5)$ takes the form.

$$T_{5D(1)}(z_1, \cdots, z_5) = \frac{1}{\lambda + \prod_{i=1}^{5}(\Delta_i + z_i)}. \quad (12)$$

B. 2D, First–order Moog ladder digital filter

The state–space formulation in 2D, related with the output $y(i_1, i_2)$ of the first–order 2D circuit implementation depicted in Fig. 4, can be expressed as follows [6], [7]:

$$\dot{\mathbf{x}}(i_1, i_2) = \mathbf{A}\mathbf{x}(i_1, i_2) + \mathbf{b}u(i_1, i_2), \quad (13)$$
$$y(i_1, i_2) = \mathbf{c}'\mathbf{x}(i_1, i_2), \quad (14)$$

where,

$$\dot{\mathbf{x}}(i, j) = \begin{bmatrix} x_1^h(i_1 + 1, i_2) \\ x_1^v(i_1, i_2 + 1) \end{bmatrix},$$
$$\mathbf{x}(i, j) = \begin{bmatrix} x_1^h(i_1, i_2) \\ x_1^v(i_1, i_2) \end{bmatrix}.$$

The following are the resulting state matrix $(\mathbf{A})$, the input vector $(\mathbf{b})$, the output vector $(\mathbf{c}')$:

$$\mathbf{A} = \begin{bmatrix} -\Delta_1 & -\lambda \\ 1 & -\Delta_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\mathbf{c}' = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The matrices $\{([\mathbf{A}]; [\mathbf{b}]; [\mathbf{c}'])\}$ of the above model (13), (14) have dimensions: $[\mathbf{A}] \to (2 \times 2), [\mathbf{b}] \to (2 \times 1), [\mathbf{c}'] \to (1 \times 2)$. The dimension of the state matrix $\mathbf{A}$ is minimal (2×2).

Utilizing the 2D z–transform on equations (13) and (14) yields,

$$T_{2D}(z_1, z_2) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b}, \quad (15)$$

where, $\mathcal{Z} = diag(z_1, z_2)$. The related transfer function in 2D $T_{2D(1)}(z_1, z_2)$ takes the following form.

$$T_{2D(1)}(z_1, z_2) = \frac{1}{\lambda + \prod_{i=1}^{2}(\Delta_i + z_i)}. \quad (16)$$

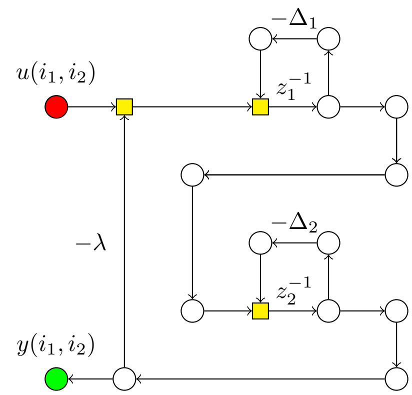

Fig. 4: 2D Moog ladder structured digital filter

The above, (16), can be represented analytically as:

$$T_{2D(1)}(z_1, z_2) = 1/[z_1 z_2 + z_1\Delta_2 + z_2\Delta_1 + (\Delta_1\Delta_2 + \lambda)].$$

The new findings obtained in this work, which pertain to five dimensions, can simply be applied to lower dimensions as well. For example using $z_1 = z_2 = z$, the above first–order 2D transfer function can represent a second–order 1D Moog ladder transfer function as follows:

$$T_{1D(2)}(z) = 1/[(z^2 + z(\Delta_1 + \Delta_2) + (\Delta_1\Delta_2 + \lambda)].$$

VI. Conclusion

In this work, a 5D Moog ladder structured digital filter was introduced, featuring minimal circuit and state–space realizations. It is acknowledged that the conventional 1D Moog filter is essential in the music industry. The 1D Moog's filter was extended to 5D, and the resulting novel realizations and their transfer functions were provided. Furthermore, the internal characteristics of the filter were demonstrated through the use of two examples. The first example employed a 5D circuit implementation, while the subsequent example was a 2D digital filter. Also, the transfer functions were included as well as the generalized state–space realization.

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