A New 4D Lattice FIR Digital Filter

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***Abstract*—This paper presents a new four–dimensional (4D) lattice structured digital filter, having a minimal number of delay elements. This filter, besides having a minimal number of delay elements, also has an absolutely minimal state–space vector. The new finite impulse response (FIR) digital filter is characterized by a lattice structure having alternate delay element orientation. Furthermore, the transfer function coefficients of the proposed 4D filter are complements of the conventional lattice filters. The results of this paper are directly applicable to one–dimensional (1D) lattice digital filters. 4D and two–dimensional (2D) low–order examples are provided to show the features of the circuit and state–space realization structures.**

I. Introduction

After decades of extensive and successful research efforts that led to appealing theoretical results and emerging applications, multidimensional (nD) systems and signal processing have been diffused to a diverse spectrum of engineering and mathematics areas, which include: digital filtering, image processing, system theory, depth–velocity filtering, light filters, 3D virtual reality, robotic vision and medical imaging systems [1]–[7].

In this publication, a new 4D lattice FIR digital filter having alternate delay element orientation is proposed. It's circuit and state–space realizations are characterized by having an absolute minimal number of delay elements, and preserving a minimal dimension of the state–vector. Essentially, the need to provide minimal realization arises not only out of hardware requirements, but also because non-minimal realizations often cause theoretical or computational difficulties due to the absence of the fundamental theorem of algebra for polynomials with more than one–dimension [3]. Over the years, absolute minimal circuit and state–space realizations, even for two–dimensions (2D), have received substantial attention because absolute minimal realizations are not always possible, except for special cases [8]. Especially in the case of 4D circuit and state–space, absolute minimal realizations have been proposed for FIR, lattice, ladder, lattice–ladder, reverberator-based, bidirectional, and Kelly–Lochbaum junction–based digital filters [9]–[16].

In the new proposed FIR lattice digital filter transfer function, with alternate delays, an interesting property is conspicuous when it is compared with its conventional FIR lattice filter equivalent [9]. The coefficients of the transfer functions are compliments of each other with respect to all forward reflection coefficients.

The rest of the paper is arranged as follows: In section II, the related 1D two–port lattice FIR filter is given, and in section III, 4D circuit and state–space realizations are presented. In section IV, first–order 4D and 2D examples are shown respectively. The conclusion is presented in section V.

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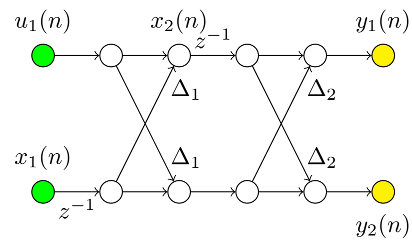

Fig. 1: 1D two–port 1D lattice FIR filter with alternate delays. $\Delta_i (i = 1, 2)$, are the forward reflection coefficients and z^{-1} is the sample time delay.

II. 1D two–port lattice FIR digital filter

A 1D two–port first–order FIR digital filter with alternate delays, Fig. 1, is described by the following equations:

$$\begin{aligned} y_1(n) &= x_2(n-1) + \Delta_2[x_1(n-1) + \Delta_1 u_1(n)] \\ y_2(n) &= x_1(n-1) + \Delta_1 u_1(n) + \Delta_2 x_2(n-1) \end{aligned}$$

or,

$$\begin{aligned} y_1(n) &= z^{-1}x_2(n) + \Delta_2 z^{-1} X_1(n) + \Delta_1\Delta_2 u_1(n) \\ y_2(n) &= z^{-1}x_1(n) + \Delta_1 u_1(n) + \Delta_2 z^{-1} x_2(n). \end{aligned}$$

Since, $x_2(n) = u_1(n) + \Delta_1 z^{-1} x_1(n)$ the above equations can be written in matrix form as:

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \begin{bmatrix} \Delta_1\Delta_2 + z^{-1} & \Delta_2 z^{-1} + \Delta_1 z^{-2} \\ \Delta_1 + \Delta_2 z^{-1} & z^{-1} + \Delta_1\Delta_2 z^{-2} \end{bmatrix} \begin{bmatrix} u_1(n) \\ x_1(n) \end{bmatrix}$$

or,

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \begin{bmatrix} \Delta_2 & z^{-1} \\ 1 & \Delta_2 z^{-1} \end{bmatrix} \begin{bmatrix} \Delta_1 & z^{-1} \\ 1 & \Delta_1 z^{-1} \end{bmatrix} \begin{bmatrix} u_1(n) \\ x_1(n) \end{bmatrix}.$$

In the following section, the above 1D digital filter will be extended to 4D and will be used to derive the state–space equations from its circuit realization.

III. 4D Circuit and State–Space Realizations

The one–dimension lattice structure, in Fig. 1, extended to 4D is given in Figs. 2 and 3. Using the Givone–Roesser state–space model [17], extended to 4D, the 4D generalized state–space model $\{\mathbf{A}, \mathbf{b}, \mathbf{c}', d\}$ of cyclic structure [18] is derived.

The subsequent goal is the derivation of the 4D generalized state–space model $\{\mathbf{A}, \mathbf{b}, \mathbf{c}', d\}$ having the well known

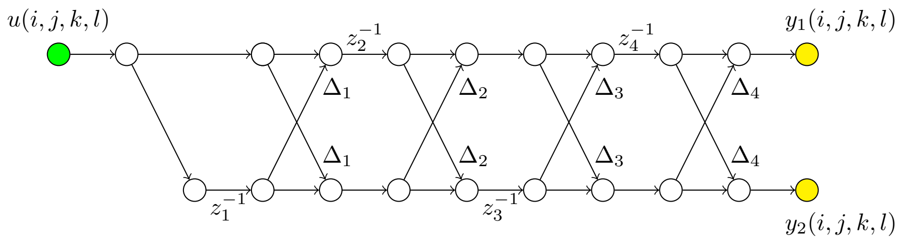

Fig. 2: 4D first–order lattice FIR digital filter with alternate delays.

structure, possessing the cyclic dimensional arrangement with respect to the four independent variables: $x_1^h(i,j,k,l)$, $x_1^v(i,j,k,l)$, $x_1^d(i,j,k,l)$, and $x_1^t(i,j,k,l)$ [17], [18].

To determine the state–space equations for the 4D model, $\{\mathbf{A}, \mathbf{b}, \mathbf{c}', d\}$, the following procedure is taken:

- Use the circuit representation, depicted in Fig. 2
- Label the outputs of the delay elements $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$ that correspond to the states of the model
- Write, by inspection, one state equation for every delay element $z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}$
- Rearrange the equations to have blocks of the state variables: $\mathbf{x}^h$, $\mathbf{x}^v$, $\mathbf{x}^d$, $\mathbf{x}^t$
- Generalize the results.

The above procedure yields the generalized 4D state–space matrix–vectors $\{\mathbf{A}, \mathbf{b}, \mathbf{c}'\}$ and the scalar d.

The overall 4D Givone–Roesser state–space system model with cyclic structure, in terms of the variables $x_1^h(i,j,k,l)$, $x_1^v(i,j,k,l)$, $x_1^d(i,j,k,l)$, $x_1^t(i,j,k,l)$, is given below [17], [18]:

$$\begin{aligned} \dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (1) \\ y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (2) \end{aligned}$$

where,

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l)) \\ x_1^t(i,j,k,l+1) \\ \cdots \\ x_{4n}^h(i+1,j,k,l) \\ x_{4n}^v(i,j+1,k,l) \\ x_{4n}^d(i,j,k+1,l)) \\ x_{4n}^t(i,j,k,l+1) \end{bmatrix}, \quad \mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l)) \\ x_1^t(i,j,k,l) \\ \cdots \\ x_{4n}^h(i,j,k,l) \\ x_{4n}^v(i,j,k,l) \\ x_{4n}^d(i,j,k,l)) \\ x_{4n}^t(i,j,k,l) \end{bmatrix}.$$

The matrices $\mathbf{A}, \mathbf{b}, \mathbf{c}'$ and the scalar d of the derived 4D state model, given in G4DL (top of the next page), have the following dimensions respectively: $4n \times 4n$, $4n \times 1$, $1 \times 4n$.

Applying the 4D z–transform on (1) and (2) yields the following 4D transfer function:

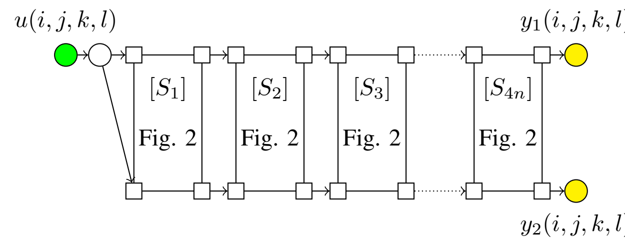

Fig. 3: 4D nth–order lattice FIR digital filter with alternate delays. Each square is a copy of the single section of Fig. 2.

$$T(z_1, z_2, z_3, z_4) = \mathbf{c}'[\mathcal{Z} - \mathbf{A}]^{-1}\mathbf{b} \quad (3)$$

where, $\mathcal{Z} = z_1\mathbf{I}_n \oplus z_2\mathbf{I}_n \oplus z_3\mathbf{I}_n \oplus z_4\mathbf{I}_n$, with $\oplus$ denoting the direct sum.

The procedure to facilitate the traversal between the state–space models of Givone–Roesser and cyclic, both extended to 4D, is provided in [18].

IV. Examples

Salient examples, simple yet illustrative of the theoretical concepts presented in this work, follow below:

A. 4D first–order lattice FIR filter with alternate delays

Selecting the output of the 4D first–order lattice filter given in Fig. 2 to be $y_1(i,j,k,l)$, the corresponding 4D state–space realization takes on the following form:

$$\begin{aligned} \dot{\mathbf{x}}(i,j,k,l) &= \mathbf{A}\mathbf{x}(i,j,k,l) + \mathbf{b}u(i,j,k,l) \quad (4) \\ y(i,j,k,l) &= \mathbf{c}'\mathbf{x}(i,j,k,l) + du(i,j,k,l) \quad (5) \end{aligned}$$

where,

$$\dot{\mathbf{x}}(i,j,k,l) = \begin{bmatrix} x_1^h(i+1,j,k,l) \\ x_1^v(i,j+1,k,l) \\ x_1^d(i,j,k+1,l) \\ x_1^t(i,j,k,l+1) \end{bmatrix}$$

$$\mathbf{x}(i,j,k,l) = \begin{bmatrix} x_1^h(i,j,k,l) \\ x_1^v(i,j,k,l) \\ x_1^d(i,j,k,l) \\ x_1^t(i,j,k,l) \end{bmatrix}$$

$$
\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \Delta_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \Delta_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \Delta_2 & 1 & \Delta_3 & 0 & 0 & 0 & \cdots & 0 \\ \Delta_2\Delta_3 & \Delta_3 & 1 & \Delta_4 & 0 & 0 & \cdots & 0 \\ \cdots & \Delta_3\Delta_4 & \Delta_4 & 1 & 0 & 0 & \cdots & 0 \\ \prod_{i=2}^{4n-3}\Delta_i & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \prod_{i=2}^{4n-2}\Delta_i & \prod_{i=3}^{4n-2}\Delta_i & \cdots & \cdots & \Delta_{4n-2} & 1 & \Delta_{4n-1} & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \Delta_1 \\ \Delta_1\Delta_2 \\ \Delta_1\Delta_2\Delta_3 \\ \cdots \\ \prod_{i=1}^{4n-3}\Delta_i \\ \prod_{i=1}^{4n-2}\Delta_i \end{bmatrix}
$$

$$
\mathbf{c}' = \begin{bmatrix} \prod_{i=2}^{4n}\Delta_i & \prod_{i=3}^{4n}\Delta_i & \prod_{i=4}^{4n}\Delta_i & \cdots & \Delta_{4n-1}\Delta_{4n} & \Delta_{4n} & 1 \end{bmatrix}, \quad d = \prod_{i=1}^{4n}\Delta_i .
$$

G4DL: Generalized 4D state–space matrix–vectors: $\mathbf{A}, \mathbf{b}, \mathbf{c}'$ and the scalar d, of (1) and (2).

4D LATTICE FIR TRANSFER FUNCTIONS

Variables	FIR (alternate)	FIR (conventional)
$z_1^{-1}z_2^{-1}z_3^{-1}z_4^{-1}$	$\Delta_1\Delta_2\Delta_3$	Δ_4
$z_1^{-1}z_2^{-1}z_3^{-1}$	$\Delta_1\Delta_2\Delta_4$	Δ_3
$z_1^{-1}z_2^{-1}z_4^{-1}$	Δ_1	$\Delta_2\Delta_3\Delta_4$
$z_1^{-1}z_2^{-1}$	$\Delta_1\Delta_3\Delta_4$	Δ_2
$z_1^{-1}z_3^{-1}z_4^{-1}$	Δ_3	$\Delta_1\Delta_2\Delta_4$
$z_1^{-1}z_3^{-1}$	Δ_4	$\Delta_1\Delta_2\Delta_3$
$z_1^{-1}z_4^{-1}$	Δ_2	$\Delta_1\Delta_3\Delta_4$
z_1^{-1}	$\Delta_2\Delta_3\Delta_4$	Δ_1
$z_2^{-1}z_3^{-1}z_4^{-1}$	$\Delta_2\Delta_3$	$\Delta_1\Delta_4$
$z_2^{-1}z_3^{-1}$	$\Delta_2\Delta_4$	$\Delta_1\Delta_3$
$z_2^{-1}z_4^{-1}$	1	$\Delta_1\Delta_2\Delta_3\Delta_4$
z_2^{-1}	$\Delta_3\Delta_4$	$\Delta_1\Delta_2$
$z_3^{-1}z_4^{-1}$	$\Delta_1\Delta_3$	$\Delta_2\Delta_4$
z_3^{-1}	$\Delta_1\Delta_4$	$\Delta_2\Delta_3$
z_4^{-1}	$\Delta_1\Delta_2$	$\Delta_3\Delta_4$
const.	$\Delta_1\Delta_2\Delta_3\Delta_4$	1

Table 1: 4D first–order lattice FIR transfer functions, one with alternate delays (center column) and one conventional (right column).

with,

$$
\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Delta_1 & 0 & 0 & 0 \\ 1 & \Delta_2 & 0 & 0 \\ \Delta_2 & 1 & \Delta_3 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \Delta_1 \\ \Delta_1\Delta_2 \end{bmatrix}
$$

$$
\mathbf{c}' = \begin{bmatrix} \Delta_2\Delta_3\Delta_4 & \Delta_3\Delta_4 & \Delta_4 & 1 \end{bmatrix}, d = \Delta_1\Delta_2\Delta_3\Delta_4 .
$$

Using (3), the corresponding 4D first–order lattice FIR transfer function with alternate delays, of (4) and (5), is presented in the Table 1 (center column).

The new proposed FIR lattice digital filter transfer function, with alternate delays (center column), has an interesting property as shown in Table 1 that is compared with its conventional lattice filter (right column) equivalent [9]. The coefficients of the transfer functions are compliments of each other with respect to all coefficients $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$.

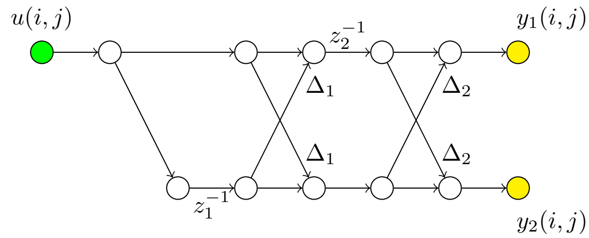

Fig. 4: 2D first–order lattice FIR filter with alternate delays.

B. *2D first–order lattice FIR filter with alternate delays*

Selecting the output of the 2D first–order lattice FIR filter given in Fig. 4 to be $y_1(i,j)$, the corresponding 2D state–space realization takes on the following form:

$$
\dot{\mathbf{x}}(i,j) = \mathbf{A}\mathbf{x}(i,j) + \mathbf{b}u(i,j) \quad (6)
$$

$$
y(i,j) = \mathbf{c}'\mathbf{x}(i,j) + du(i,j) \quad (7)
$$

where,

$$
\dot{\mathbf{x}}(i,j) = \begin{bmatrix} x_1^h(i+1,j) \\ x_1^v(i,j+1) \end{bmatrix}, \ \mathbf{x}(i,j) = \begin{bmatrix} x_1^h(i,j) \\ x_1^v(i,j) \end{bmatrix}
$$

with,

$$
\mathbf{A} = \begin{bmatrix} 0 & 0 \\ \Delta_1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
\mathbf{c}' = \begin{bmatrix} \Delta_2 & 1 \end{bmatrix}, \quad d = \Delta_1\Delta_2.
$$

Using (3) in the 2D sense, the corresponding 2D transfer function $T_1(z_1^{-1}, z_2^{-1})$ of (6) and (7) is presented in the following equation:

$$
T_1(z_1^{-1}, z_2^{-1}) = \Delta_1 z_1^{-1} z_2^{-1} + \Delta_2 z_1^{-1} + z_2^{-1} + \Delta_1\Delta_2. \quad (8)
$$

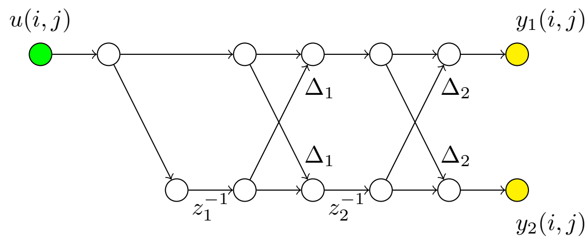

Fig. 5: 2D first–order lattice FIR conventional digital filter.

C. *2D first–order lattice FIR conventional filter*

Selecting the output of the 2D first–order lattice FIR conventional filter [9], [19], which is given in Fig. 5 to be $y_1(i,j)$, the corresponding 2D state–space realization takes on the following form:

$$\begin{aligned} \dot{\mathbf{x}}(i,j) &= \mathbf{A}\mathbf{x}(i,j) + \mathbf{b}u(i,j) & (9)\\ y(i,j) &= \mathbf{c}'\mathbf{x}(i,j) + du(i,j) & (10) \end{aligned}$$

where,

$$\dot{\mathbf{x}}(i,j) = \begin{bmatrix} x_1^h(i+1,j) \\ x_1^v(i,j+1) \end{bmatrix}, \ \mathbf{x}(i,j) = \begin{bmatrix} x_1^h(i,j) \\ x_1^v(i,j) \end{bmatrix}$$

with,

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & 0 \\ \Delta_1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ \Delta_1 \end{bmatrix} \\ \mathbf{c}' &= \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}, \quad d = 1. \end{aligned}$$

Using (3) in the 2D sense, the corresponding 2D transfer function $T_2(z_1^{-1}, z_2^{-1})$ of (9) and (10) is presented in the following equation:

$$T_2(z_1^{-1}, z_2^{-1}) = \Delta_2 z_1^{-1} z_2^{-1} + \Delta_1 z_1^{-1} + \Delta_1 \Delta_2 z_2^{-1} + 1. \quad (11)$$

Transfer functions (8) and (11) are integrated in the following Table 2 for comparison purposes.

2D LATTICE FIR TRANSFER FUNCTIONS

Variables	FIR (alternate)	FIR (conventional)
$z_1^{-1} z_2^{-1}$	Δ_1	Δ_2
z_1^{-1}	Δ_2	Δ_1
z_2^{-1}	1	$\Delta_1\Delta_2$
$const.$	$\Delta_1\Delta_2$	1

Table 2: 2D first–order lattice FIR transfer functions, one with alternate delays (center column) and one conventional (right column).

The coefficients of the two 2D lattice FIR lattice transfer functions, Table 2, one with alternate delays (center column) and one conventional (right column), are compliments of each other with respect to all coefficients (Δ_1, Δ_2), as in the 4D case.

V. Conclusion

This paper discusses the circuit and state-space realizations of a new lattice FIR filter with alternate delays. The proposed 4D circuit realization uses a minimal number of delays $(4n)$. Additionally, the dimension of the 4D state–space vector, being $4n$ is also absolute minimal. The presented 4D and 2D examples show clearly the state–space filter structure and their corresponding transfer functions. It is noted that the coefficients of the new filter's transfer functions, with alternate delays, are complements of the conventional lattice filter coefficients. The results of this paper are directly applicable to 1D lattice FIR digital filters.

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